Interpolation systems for ground proofs

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What is interpolation?

- Formulæ A and B such that $A \vdash B$
- An interpolant I is a formula that lies between A and B:
  - Derivability: $A \vdash I$ and $I \vdash B$
  - Signature: I made of symbols common to A and B where symbol means predicate, function, constant symbol

Trivial cases

- All symbols of A appear in B: then A itself is the interpolant
- All symbols of B appear in A: then B itself is the interpolant
- Assume that at least one has at least one symbol that does not appear in the other

Craig's Interpolation Theorem (1957)

Closed formula: all variables are quantified (aka: sentence)

- A and B closed formulæ with at least one predicate symbol in common
- Interpolant I exists and it is also a closed formula
- No predicate symbol in common: either A is unsatisfiable and I is $\perp$ or B is valid and I is $\top$

Clausal theorem proving: A and B are sets of clauses

Proofs by refutation: reverse interpolant

- A and B inconsistent: $A, B \vdash \perp$
- Then $A \vdash I$ and $B, I \vdash \perp$
- All symbols in I common to A and B

Reverse interpolant of (A, B): interpolant of $(A, \neg B)$
because $A, B \vdash \perp$ means $A \vdash \neg B$ and $B, I \vdash \perp$ means $I \vdash \neg B$

In refutational settings we say interpolant for reverse interpolant

Reasoning modulo theory $\mathcal{T}$

- $\vdash_{\mathcal{T}}$ in place of $\vdash$
- All uninterpreted symbols in I common to A and B
- No restrictions on interpreted symbols

Example in propositional logic

$A = \{a \vee e,\ \neg a \vee b,\ \neg a \vee c\} \quad B = \{\neg b \vee \neg c \vee d,\ \neg d,\ \neg e\}$

1. $a \vee e$ resolves with $\neg e$ to yield a
2. a resolves with $\neg a \vee c$ to yield c
3. a resolves with $\neg a \vee b$ to yield b
4. b resolves with $\neg b \vee \neg c \vee d$ to yield $\neg c \vee d$
5. c resolves with $\neg c \vee d$ to yield d
6. d resolves with $\neg d$ to yield $\Box$

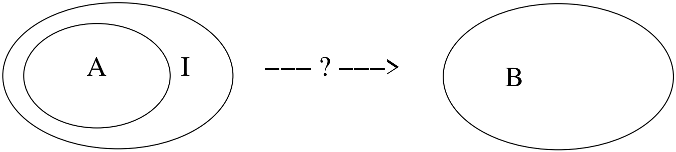
Interpolant I: $(e \vee b) \wedge (e \vee c) \equiv e \vee (b \wedge c)$

Why interpolation?

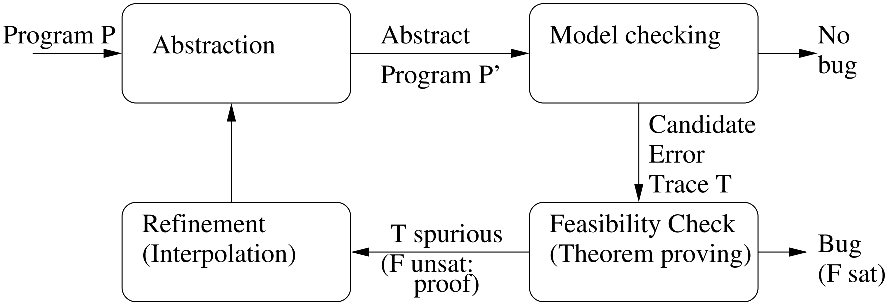
- Interpolant is a formula in between formulæ
- Formulæ represent states that satisfy them
- States of an automaton, of a transition system, of a program
- Interpolant may give information on intermediate states

Image computation in model checking

- Transition system with transition relation
- Forward reachability: computing images
- Backward reachability: computing pre-images
- Interpolant: over-approximation of an image/pre-image
- Interpolation to accelerate convergence towards fixed point



Abstraction refinement in software model checking



$F = A \cup B$; add predicates from interpolant I of (A, B): exclude T

Automated invariant generation

- Loop: *pre* `while C do T` *post*
  - $\forall s.\ pre[s] \supset I(s)$
  - $\forall s, s'.\ I(s) \wedge C[s] \wedge T[s, s'] \supset I(s')$
  - $\forall s.\ I(s) \wedge \neg C[s] \supset post(s)$
- Invariant I made of symbols common to *pre* and *post*; no symbols local to the loop body `T`

- A: k-unfolding of loop; B: post-condition violated
- $A, B \vdash \bot$
- Interpolant of (A, B): candidate invariant

Several approaches to interpolation

- Building interpolation into satisfiability procedures (e.g., congruence closure) [Fuchs, Goel, Grundy, Krstić, Tinelli 2012]
- Locality based [Sofronie-Stokkermans 2008]
- Via Horn clause reasoning [Gupta, Popeea, Rybalchenko 2011], [Rümmer, Hojjat, Kuncak 2013]
- Meta-rules based approach [Bruttomesso, Ghilardi, Ranise 2012], [Bruttomesso, Ghilardi, Ranise 2014]
- Inductive approach: by structural induction on the refutation

Terminology for interpolation: Colors

Uninterpreted symbol:

- A-colored: occurs in A and not in B
- B-colored: occurs in B and not in A
- Transparent: occurs in both

Alternative terminology: A-local, B-local, global

Terminology for interpolation: Colors

Ground term/literal/clause:

- All transparent symbols: transparent
- A-colored (at least one) and transparent symbols: A-colored
- B-colored (at least one) and transparent symbols: B-colored
- Otherwise: AB-mixed

Interpolation system

- A and B sets of clauses
- Given: a refutation of $A \cup B$
- Interpolation system: extracts interpolant of (A, B)
- How? Computing a partial interpolant $PI(C)$ for each clause C in refutation
- Defined in such a way that $PI(\Box)$ is interpolant of (A, B)

Partial interpolant

- Clause C in refutation of $A \cup B$
- $A \wedge B \vdash C$
- $A \wedge B \vdash C \vee C$
- $A \wedge \neg C \vdash \neg B \vee C$
- Interpolant of $A \wedge \neg C$ and $\neg B \vee C$
- Reverse interpolant of $A \wedge \neg C$ and $B \wedge \neg C$
- The signatures of $A \wedge \neg C$ and $B \wedge \neg C$ are not necessarily those of A and B unless C is transparent
- Use projections

Symmetric projections

C: disjunction (conjunction) of literals

- $C|_A$: A-colored and transparent literals
- $C|_B$: B-colored and transparent literals
- $C|_{A,B}$: transparent literals
- $\bot$ ($\top$) if empty

If C has no AB-mixed literals: $C = C|_A \vee C|_B$

Asymmetric projections

C: disjunction (conjunction) of literals

- $C\backslash_B = C|_A \setminus C|_{A,B}$ (A-colored only)
- $C \downarrow_B = C|_B$ (transparent go with B-colored)

If C has no AB-mixed literals: $C = C \backslash_B \vee C \downarrow_B$

Partial interpolant

- Clause C in refutation of $A \cup B$
- Partial interpolant $PI(C)$: interpolant of
  $A \wedge \neg(C|_A)$ and
  $B \wedge \neg(C|_B)$
- If C is $\Box$: $PI(C)$ interpolant of (A, B)
- Requirements:
  - $A \wedge \neg(C|_A) \vdash PI(C)$
  - $B \wedge \neg(C|_B) \wedge PI(C) \vdash \perp$
  - $PI(C)$ transparent
- Or as above with asymmetric projections

Complete interpolation system

An interpolation system is complete for an inference system if

- For all sets of clauses A and B such that $A \cup B$ is unsatisfiable
- For all refutations of $A \cup B$ by the inference system

It generates an interpolant of (A, B)

There may be more than one

Inductive approach to interpolation

- The interpolation system is defined inductively
- By defining the partial interpolant of the consequence given the partial interpolants of the premises
- For all generative inference rules (e.g., superposition, simplification, not subsumption)
- Prove complete:
  show that its partial interpolants are indeed such

Interpolation for propositional resolution

- DPLL-CDCL
- Inference system Γ with resolution, superposition, simplification, subsumption ...

- If given a problem in propositional logic
- Both generate proof by resolution

Propositional interpolation systems

- Literals in proof are input literals
- Input literals are either A-colored or B-colored or transparent
- No AB-mixed literals

The HKPYM interpolation system

C clause in refutation of $A \cup B$ by propositional resolution:

- $C \in A$: $PI(C) = \bot$
- $C \in B$: $PI(C) = \top$
- $C \vee D$ propositional resolvent of $p_1 : C \vee L$ and $p_2 : D \vee \neg L$:
  - L A-colored: $PI(C \vee D) = PI(p_1) \vee PI(p_2)$
  - L B-colored: $PI(C \vee D) = PI(p_1) \wedge PI(p_2)$
  - L transparent: $PI(C \vee D) = (L \vee PI(p_1)) \wedge (\neg L \vee PI(p_2))$

Symmetric projections

[Huang 1995] [Krajíček 1997] [Pudlàk 1997] [Yorsh, Musuvathi 2005]

Example with HKPYM

$A = \{a \vee e,\ \neg a \vee b,\ \neg a \vee c\} \quad B = \{\neg b \vee \neg c \vee d,\ \neg d,\ \neg e\}$

1. $a \vee e$ $[\bot]$ resolves with $\neg e$ $[\top]$ to yield a $[e]$:
   $PI(a) = (e \vee \bot) \wedge (\neg e \vee \top) = e$
2. a $[e]$ resolves with $\neg a \vee c$ $[\bot]$ to yield c $[e]$: $PI(c) = e \vee \bot = e$
3. a $[e]$ resolves with $\neg a \vee b$ $[\bot]$ to yield b $[e]$: $PI(b) = e \vee \bot = e$
4. b $[e]$ resolves with $\neg b \vee \neg c \vee d$ $[\top]$ to yield $\neg c \vee d$ $[b \vee e]$:
   $PI(\neg c \vee d) = (b \vee e) \wedge (\neg b \vee \top) = b \vee e$
5. c $[e]$ resolves with $\neg c \vee d$ $[b \vee e]$ to yield d $[e \vee (c \wedge b)]$:
   $PI(d) = (c \vee e) \wedge (\neg c \vee b \vee e) = e \vee (c \wedge b)$
6. d $[e \vee (c \wedge b)]$ resolves with $\neg d$ $[\top]$ to yield $\Box$ $[e \vee (c \wedge b)]$:
   $PI(\Box) = (e \vee (c \wedge b)) \wedge \top = e \vee (c \wedge b)$

The MM interpolation system

C clause in refutation of $A \cup B$ by propositional resolution:

- $C \in A$: $PI(C) = C|_{A,B}$
- $C \in B$: $PI(C) = \top$
- $C \vee D$ propositional resolvent of $p_1 : C \vee L$ and $p_2 : D \vee \neg L$:
  - L A-colored: $PI(C \vee D) = PI(p_1) \vee PI(p_2)$
  - L B-colored or transparent: $PI(C \vee D) = PI(p_1) \wedge PI(p_2)$

Asymmetric projections

[McMillan 2003]

Example with MM

$A = \{a \vee e,\ \neg a \vee b,\ \neg a \vee c\}$ $\quad B = \{\neg b \vee \neg c \vee d,\ \neg d,\ \neg e\}$

1. $a \vee e$ $[e]$ resolves with $\neg e$ $[\top]$ to yield a $[e]$: $PI(a) = e \wedge \top = e$
2. a $[e]$ resolves with $\neg a \vee c$ $[c]$ to yield c $[e \vee c]$: $PI(c) = e \vee c$
3. a $[e]$ resolves with $\neg a \vee b$ $[b]$ to yield b $[e \vee b]$: $PI(b) = e \vee b$
4. b $[e \vee b]$ resolves with $\neg b \vee \neg c \vee d$ $[\top]$ to yield $\neg c \vee d$ $[e \vee b]$: $PI(\neg c \vee d) = (e \vee b) \wedge \top = e \vee b$
5. c $[e \vee c]$ resolves with $\neg c \vee d$ $[e \vee b]$ to yield d $[e \vee (c \wedge b)]$: $PI(d) = (e \vee c) \wedge (e \vee b) = e \vee (c \wedge b)$
6. d $[e \vee (c \wedge b)]$ resolves with $\neg d$ $[\top]$ to yield $\Box$ $[e \vee (c \wedge b)]$: $PI(\Box) = (e \vee (c \wedge b)) \wedge \top = e \vee (c \wedge b)$

Comparison of HKPYM and MM

- In this example the final interpolant is the same, although at each step the HKPYM partial interpolant implies the MM partial interpolant
- In general: MM interpolants imply HKPYM interpolants [D'Silva, Kroening, Purandare, Weissenbacher 2010]
- But there is no general result as to whether weaker or stronger is preferable

Equality changes the picture ...

- Propositional logic: no AB-mixed literals and colors are stable
- Equality: what if AB-mixed equality $t_a \simeq t_b$ is derived?
  t_a: A-colored ground term; t_b: B-colored ground term
- Congruence closure: t_a and t_b representatives of singly-colored classes: merge: one of them should become transparent
- Rewriting: t_a and t_b in normal form, $t_a \succ t_b$:
  rewrite t_a as t_b; t_b should become transparent
- A-colored/B-colored/transparent cannot change dynamically!

Equality-interpolating theory

- $\mathcal{T}$: convex theory
- (A, B): there exist transparent ground terms
- If $A \wedge B \models_{\mathcal{T}} t_a \simeq t_b$
  t_a: A-colored ground term and t_b: B-colored ground term
- Then $A \wedge B \models_{\mathcal{T}} t_a \simeq t \wedge t_b \simeq t$ for some transparent ground term t called equality-interpolating term

Congruence closure: t representative of the new congruence class

[Yorsh, Musuvathi 2005]

Separating ordering

Ordering $\succ$ on terms and literals:
separating if $s \succ r$ whenever r is transparent and s is not

Rewriting: t_a and t_b rewritten to t

[McMillan 2008], [Kovàcs, Voronkov 2009]

Separating implies no AB-mixed literals

- Γ: inference system with resolution, superposition, simplification, subsumption ...
- Lemma: If the ordering $\succ$ is separating, ground Γ-refutations contain no AB-mixed literals
  - $s \simeq r$ and $l[s]$ not AB-mixed, and $s \succ r$
  - either s and r same color or r transparent
  - $l[r]$ not AB-mixed

EUF is equality-interpolating

- Theorem: The quantifier-free fragment of the theory of equality is equality-interpolating
  - Γ with $\succ$ separating ordering
  - (A, B): there exist transparent ground terms
  - If $A \wedge B \models t_a \simeq t_b$
  - $A \cup B \cup \{t_a \not\simeq t_b\} \vdash_\Gamma \bot$ by refutational completeness of Γ
  - No AB-mixed equalities as $\succ$ is separating
  - Valley proof $t_a \stackrel{*}{\rightarrow} t \stackrel{*}{\leftarrow} t_b$ contains at least a transparent term
  - t must be transparent

Other convex equality-interpolating theories

- Non-empty lists
- Linear rational arithmetic:
  - $A \wedge B \supset a \simeq b$
  - $A \wedge B \supset a \leq b \wedge b \leq a$
  - $\exists\ t_1$ such that $A \wedge B \supset a \leq t_1 \leq b$
  - $\exists\ t_2$ such that $A \wedge B \supset b \leq t_2 \leq a$
  - $A \wedge B \supset a \simeq t_1 \simeq t_2 \simeq b$

[Yorsh, Musuvathi 2005]

Equality sharing aka Nelson-Oppen method

$\mathcal{T}$-satisfiability procedure for $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i$

- Disjoint, convex, equality-interpolating theories
- Equipped with $\mathcal{T}_i$-satisfiability procedure $\mathcal{Q}_i$ that generate equality-interpolating terms, proofs, and $\mathcal{T}_i$-interpolants
- S input set of ground $\mathcal{T}$-literals
- Partition $S = A \cup B$ and separation $S_1, \ldots, S_n$ are orthogonal: new free constants inherit the color of the term they replace, since there are no AB-mixed input terms

Interpolation in equality sharing

- Each $\mathcal{Q}_i$ takes as input $S_i = A_i \cup B_i$ and deals with $A_i \cup B_i \cup K$ where K contains the propagated equalities
- Equality-interpolating: K contains no AB-mixed equalities
- The proof by equality sharing contains no AB-mixed literals
- What is the partial interpolant for a propagated equality?
- Theory-specific partial interpolant

Theory-specific partial interpolant

- Propagated literal: $A_i \cup B_i \cup K \vdash_{\mathcal{T}_i} L$
  where L is either an equality or $\Box$
- Interpolation wrt partition (A', B') of $A_i \cup B_i \cup K$
  $A' = A_i \cup K \backslash_B$
  $B' = B_i \cup K \downarrow_B$
- $PI^i_{(A',B')}(L)$ is the $\mathcal{T}_i$-interpolant of
  $(A' \wedge \neg(L\backslash_B), B' \wedge \neg(L \downarrow_B))$

[Yorsh, Musuvathi 2005]

The YM interpolation system

C unit clause in refutation of $A \cup B$ by equality sharing

- $C \in A$: $PI(C) = \bot$ $\quad$ $C \in B$: $PI(C) = \top$
- C derived as $A_i \cup B_i \cup K \vdash_{\mathcal{T}_i} C$:
  $PI(C) = (PI^i_{(A',B')}(C) \vee \bigvee_{L \in A'} PI(L)) \wedge \bigwedge_{L \in B'} PI(L)$

If $K = \emptyset$ (only one theory or C does not depend on propagated equalities): $PI(C) = PI^i_{(A',B')}(C)$

Example in theory combination

$$A = \{f(x_1) + x_2 \simeq x_3, \quad f(y_1) + y_2 \simeq y_3, \quad y_1 \leq x_1\}$$

$$B = \{x_2 \simeq g(b), \quad y_2 \simeq g(b), \ x_1 \leq y_1, \quad x_3 < y_3\}$$

Let EUF be $\mathcal{T}_1$ with procedure $\mathcal{Q}_1$ and
LRA be $\mathcal{T}_2$ with procedure $\mathcal{Q}_2$

[Yorsh, Musuvathi 2005]

Example after separation

$A_1 = \{a_1 \simeq f(x_1),\quad a_2 \simeq f(y_1)\}$

$A_2 = \{a_1 + x_2 \simeq x_3,\quad a_2 + y_2 \simeq y_3,\quad y_1 \leq x_1\}$

$B_1 = \{x_2 \simeq g(b),\quad y_2 \simeq g(b)\}$

$B_2 = \{x_1 \leq y_1,\quad x_3 < y_3\}$

Shared constants: $V = \{a_1, x_1, a_2, y_1, x_2, y_2\}$

$\{f, a_1, a_2\}$ are A-colored

$\{g, b\}$ are B-colored

$\{x_1, y_1, x_2, y_2, x_3, y_3\}$ are transparent

Example: first proof step

- Q_2 deduces $x_1 \simeq y_1$ from $y_1 \leq x_1$ $[\perp]$ and $x_1 \leq y_1$ $[\top]$
- $x_1, y_1 \in V$: $x_1 \simeq y_1$ is propagated
- $A' = A_2$ and $B' = B_2$ since $K = \emptyset$
- $A' \wedge \neg((x_1 \simeq y_1)\backslash_B) = A_2 \wedge \top = A_2$
  $B' \wedge \neg((x_1 \simeq y_1)\downarrow_B) = B_2 \cup \{x_1 \not\simeq y_1\}$
- $PI^2_{(A',B')}(x_1 \simeq y_1) = y_1 \leq x_1$
  which follows from $y_1 \leq x_1 \in A_2$ and is $\mathcal{T}_2$-inconsistent with $\{x_1 \leq y_1,\ x_1 \not\simeq y_1\}$ where $x_1 \leq y_1 \in B_2$
- $PI(x_1 \simeq y_1) = y_1 \leq x_1$

Example: second proof step

- Q_1 deduces $a_1 \simeq a_2$ from $a_1 \simeq f(x_1)$ $[\perp]$, $a_2 \simeq f(y_1)$ $[\perp]$ and $x_1 \simeq y_1$ $[y_1 \leq x_1]$
- $a_1, a_2 \in V$: $a_1 \simeq a_2$ is propagated
- $A' = A_1$ and $B' = B_1 \cup \{x_1 \simeq y_1\}$ since $K = \{x_1 \simeq y_1\}$
- $A' \wedge \neg((a_1 \simeq a_2)\backslash_B) = A_1 \cup \{a_1 \not\simeq a_2\}$
  $B' \wedge \neg((a_1 \simeq a_2) \downarrow_B) = B_1 \cup \{x_1 \simeq y_1\}$
- $PI^1_{(A',B')}(a_1 \simeq a_2) = x_1 \not\simeq y_1$
  which follows from $\{a_1 \simeq f(x_1),\ a_2 \simeq f(y_1),\ a_1 \not\simeq a_2\}$ and is inconsistent with $\{x_1 \simeq y_1\}$
- $PI(a_1 \simeq a_2) = (x_1 \not\simeq y_1 \vee \perp) \wedge y_1 \leq x_1 = y_1 < x_1$

Example: third proof step

- Q_1 deduces $x_2 \simeq y_2$ from $x_2 \simeq g(b)$ $[\top]$ and $y_2 \simeq g(b)$ $[\top]$
- $x_2, y_2 \in V$: $x_2 \simeq y_2$ is propagated
- $A' = A_1$ and $B' = B_1$ since $K = \emptyset$
- $A' \wedge \neg((x_2 \simeq y_2)\backslash_B) = A_1 \wedge \top = A_1$
  $B' \wedge \neg((x_2 \simeq y_2) \downarrow_B) = B_1 \cup \{x_2 \not\simeq y_2\}$
- $PI^1_{(A',B')}(x_2 \simeq y_2) = \top$
  because $B_1 \cup \{x_2 \not\simeq y_2\}$ is $\mathcal{T}_1$-inconsistent
- $PI(x_2 \simeq y_2) = \top$

Example: fourth proof step

- Q_2 deduces $\Box$ from $a_1 + x_2 \simeq x_3\ [\bot]$, $a_2 + y_2 \simeq y_3\ [\bot]$, $x_3 < y_3\ [\top]$, $a_1 \simeq a_2\ [y_1 < x_1]$ and $x_2 \simeq y_2\ [\top]$
- $A' = A_2 \cup \{a_1 \simeq a_2\}$ and $B' = B_2 \cup \{x_2 \simeq y_2\}$ as $K = \{a_1 \simeq a_2, x_2 \simeq y_2\}$
- $A' \wedge \neg((\Box)\backslash_B) = A_2 \cup \{a_1 \simeq a_2\} \wedge \top = A_2 \cup \{a_1 \simeq a_2\}$
  $B' \wedge \neg((\Box)\downarrow_B) = B_2 \cup \{x_2 \simeq y_2\} \wedge \top = B_2 \cup \{x_2 \simeq y_2\}$
- $PI^2_{(A',B')}(\Box) = x_3 - x_2 \simeq y_3 - y_2$
  because $\{a_1 + x_2 \simeq x_3,\ a_2 + y_2 \simeq y_3,\ a_1 \simeq a_2\}$ entail $x_3 - x_2 \simeq y_3 - y_2$ which is $\mathcal{T}_2$-inconsistent with $\{x_3 < y_3,\ x_2 \simeq y_2\}$ where $x_3 < y_3 \in B_2$
- $PI(\Box) = (x_3 - x_2 \simeq y_3 - y_2 \vee y_1 < x_1) \wedge \top = x_3 - x_2 \simeq y_3 - y_2 \vee y_1 < x_1$

Interpolation in DPLL($\mathcal{T}$)

- $A \cup B$ set of ground $\mathcal{T}$-clauses
- DPLL($\mathcal{T}$)-refutation of $A \cup B$: propositional resolution + $\mathcal{T}$-lemmas ($\mathcal{T}$-conflict clauses are $\mathcal{T}$-lemmas)
- If clause C is a $\mathcal{T}$-lemma, $\neg C$ is a $\mathcal{T}$-unsatisfiable set of ground $\mathcal{T}$-literals
- No *AB*-mixed literals: $\neg C = (\neg C) \setminus_B \wedge (\neg C) \downarrow_B$
- The $\mathcal{T}$-interpolant of $((\neg C)\setminus_B, (\neg C) \downarrow_B)$ computed by YM provides partial interpolant of C in DPLL($\mathcal{T}$)-refutation

HKPYM–T and MM–T interpolation systems

Add one case to either HKPYM or MM:

- C is a $\mathcal{T}$-lemma:
  $PI(C)$ is $\mathcal{T}$-interpolant of $((\neg C)\backslash_B, (\neg C)\downarrow_B)$ extracted by YM from $\neg C \vdash_{\mathcal{T}} \bot$

Completeness: from that of HKPYM or MM and YM

[Yorsh and Musuvathi 2005]

Why interpolation for superposition?

- Superposition-based decision procedures
- DPLL($\Gamma+\mathcal{T}$): DPLL($\mathcal{T}$) with superposition (Γ) integrated for a fully automated treatment of quantifiers

Interpolation system GΓI

C clause in ground Γ-refutation of $A \cup B$:

- Base cases and resolution: same as in HKPYM
- $c\colon C \vee l[r] \vee D$ generated from $p_1\colon C \vee s \simeq r$ and $p_2\colon l[s] \vee D$
  - $s \simeq r$ A-colored: $PI(c) = PI(p_1) \vee PI(p_2)$
  - $s \simeq r$ B-colored: $PI(c) = PI(p_1) \wedge PI(p_2)$
  - $s \simeq r$ transparent: $PI(c) = (s \simeq r \vee PI(p_1)) \wedge (s \not\simeq r \vee PI(p_2))$
- Superposition into equational literal and Simplification: same

Example with superposition

$A = \{P(c),\ \neg P(e)\} \qquad B = \{c \simeq e\} \qquad c \succ e$

P is A-colored, c and e are transparent

1. $c \simeq e$ $[\top]$ simplifies $P(c)$ $[\bot]$ into $P(e)$ $[c \not\simeq e]$
   $PI(P(e)) = (c \simeq e \vee \top) \wedge (c \not\simeq e \vee \bot) = c \not\simeq e$
2. $\neg P(e)$ $[\bot]$ resolves with $P(e)$ $[c \not\simeq e]$ to yield $\Box$ $[c \not\simeq e]$
   $PI(\Box) = \bot \vee c \not\simeq e = c \not\simeq e$

Another example with superposition

$A = \{Q(f(a)),\ f(a) \simeq c\}$ $\qquad B = \{\neg Q(f(b)),\ f(b) \simeq c\}$

a is A-colored, b is B-colored, all other symbols are transparent

1. $f(a) \simeq c$ $[\bot]$ simplifies $Q(f(a))$ $[\bot]$ into $Q(c)$ $[\bot]$
   where $f(a) \succ c$ in any separating ordering
   $PI(Q(c)) = \bot \vee \bot = \bot$
2. $f(b) \simeq c$ $[\top]$ simplifies $\neg Q(f(b))$ $[\top]$ into $\neg Q(c)$ $[\top]$
   where $f(b) \succ c$ in any separating ordering
   $PI(\neg Q(c)) = \top \wedge \top = \top$
3. $Q(c)$ $[\bot]$ resolves with $\neg Q(c)$ $[\top]$ to yield $\square$ $[Q(c)]$
   $PI(\square) = (Q(c) \vee \bot) \wedge (\neg Q(c) \vee \top) = Q(c)$

Completeness

- Theorem: If the ordering is separating, $G\Gamma I$ is a complete interpolation system for ground Γ-refutations
- The proof shows that the partial interpolants built by $G\Gamma I$ satisfy the requirements for partial interpolants.

Summary

- Survey of interpolation systems for ground refutations:
  - Unified framework of definitions for interpolation
  - Interpolation systems for propositional resolution
  - Interpolation and equality: connecting equality-interpolating theory and separating ordering
  - Interpolation system for equality sharing
  - Interpolation systems for DPLL($\mathcal{T}$)
- A complete interpolation system for ground refutations by superposition

References

- Maria Paola Bonacina and Moa Johansson. Interpolation systems for ground proofs in automated deduction: a survey. Journal of Automated Reasoning, 54(4):353-390, 2015 [providing 89 references]
- Maria Paola Bonacina and Moa Johansson. Towards interpolation in an SMT solver with integrated superposition. 9th SMT Workshop, Snowbird, Utah, USA, July 2011; TR UCB/EECS-2011-80, 9-18, 2011
- Maria Paola Bonacina and Moa Johansson. On interpolation in decision procedures. In Proc. of the 20th TABLEAUX Conference, Bern, Switzerland, July 2011; Springer, LNAI 6793, 1–16, 2011

Discussion

- Generality: interpolants for more logics, theories, inference systems
- Quality: better interpolants; stronger? weaker? shorter?
- Non-ground proofs, non-convex theories?
  Two-stage approach:
  Maria Paola Bonacina and Moa Johansson. On interpolation in automated theorem proving. Journal of Automated Reasoning, 54(1):69-97, 2015