The CDSAT Paradigm for Theory Combination in SMT

(Based on joint work with S. Graham-Lengrand and N. Shankar)

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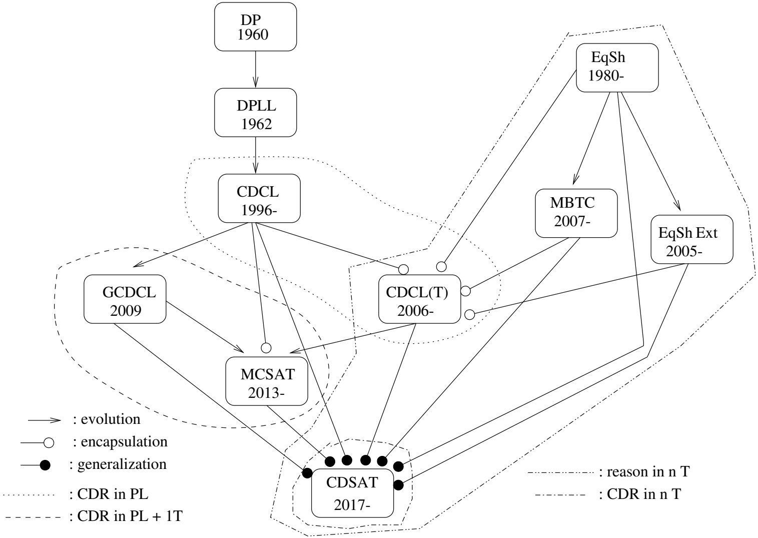
Introduction

- SMT: Satisfiability Modulo Theory
- CDSAT (Conflict-Driven Satisfiability)
  - Allows conflict-driven reasoning in a theory union
  - Generalizes previous conflict-driven procedures
- Quantifier-free input problem: set of ground clauses

Conflict-driven decision procedures

- Search for a model:
  - Decide assignments of values to terms
  - Propagate consequences of assignments
  - Conflict: contradiction
- Perform inferences only to solve conflicts (or reach `unsat`):
  - Explain conflict by inferences
    (steps towards a possible refutation)
  - Learn generated lemma that excludes current assignment:
    avoid hitting same conflict
  - Solve conflict by amending assignment to satisfy lemma
- Search and inferences guide each other:
  - Search focuses inferences on conflicts
  - Inferences allow search to escape dead-end's

The big picture



CDSAT: most general conflict-driven reasoning procedure

- Theory $\mathcal{T} = \bigcup_{k=1}^{n} \mathcal{T}_k$: predicate-sharing theories
  Disjoint if $\simeq$ is the only shared symbol
- Decide satisfiability modulo theory and assignment (SMA): input may include initial assignment
  - Boolean assignment: $L \leftarrow \mathrm{true}$ (Boolean value)
  - First-order assignment: $x \leftarrow 3$ (non-Boolean value)
- Answer sat if there exists satisfying assignment including initial one, unsat otherwise
- Initial assignment is relevant for parallelization, optimization as satisfiability, quantified satisfiability (QSMA)

CDSAT: most general conflict-driven reasoning procedure

- Transition system: transition rules (e.g., Decide, Deduce)
- Coordinates theory modules:
  $\mathcal{T}_k$-inference system + finite local basis
- Offers conflict-driven control accommodating also non-conflict-driven procedures (black-box modules)
- The modules collaborate as peers
  on a shared trail Γ containing the current assignment
- Each module offers decisions and deductions
  propagation, conflict detection, explanation
- Sound, complete, terminating under suitable hypotheses
  - Finite global basis from the local ones for termination

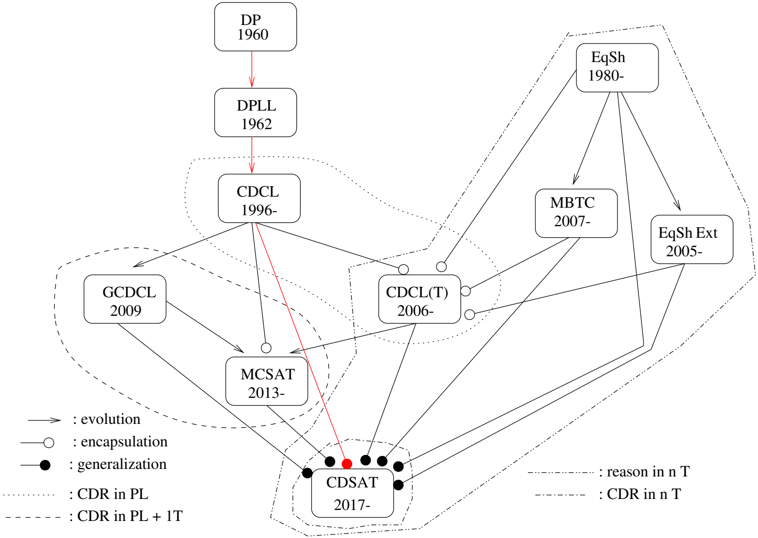
Assignments take center stage

- Assignments of values to terms:
  $(x > 1) \leftarrow \text{false}$, $((x > 1) \vee (y < 0)) \leftarrow \text{true}$,
  $(\text{store}(a, i, v) \simeq b) \leftarrow \text{true}$, $y \leftarrow \sqrt{2}$, $\text{select}(a, j) \leftarrow 3$
- Term and value have the same sort
- Formulas are Boolean terms (sort prop)
- Plausible assignment: does not contain $L \leftarrow \text{true}$ and $L \leftarrow \text{false}$
- Terms and values are kept separate:
  term only on the left, value only on the right of an assignment
- $\text{select}(a, j) \leftarrow 3$ cannot be replaced by $select(a, j) \simeq 3$:
  a value is not a term, is not in the signature
- What are values?

Theory extensions to define values

- From theory $\mathcal{T}_k$ to theory extension $\mathcal{T}_k^+$:
  - Add new constant symbols (and possibly axioms)
  - E.g.: add a constant symbol for every number (integers, rationals, algebraic reals)
    $\sqrt{2}$ is a constant symbol interpreted as $\sqrt{2}$
  - All $\mathcal{T}_k^+$'s add true and false (all $\mathcal{T}_k$'s have sort prop)
  - Trivial if it adds only true and false
- Values in assignments are these constant symbols: $\mathcal{T}_k$-values
- $\mathcal{T}_k$-assignment: assigns $\mathcal{T}_k$-values
- Conservative theory extension: $\mathcal{T}_k^+$-unsatisfiable implies $\mathcal{T}_k$-unsatisfiable

The big picture: propositional reasoning



Conflict-driven propositional satisfiability: CDCL

[Marques Silva, Sakallah: ICCAD 1996, IEEE TOC 1999]

- Candidate partial model represented as a trail Γ of Boolean assignments (stack)
- Decision: add L to Γ if $L \notin \Gamma$ and $\overline{L} \notin \Gamma$
  Every decision opens new level on Γ
- Unit propagation detects implied literals and conflict clauses
- Resolution to explain conflict: learn resolvent C
- Backjump away from conflict to a state that satisfies C
- First assertion clause (or 1UIP) heuristic

CDSAT reduces to CDCL if Bool is the only theory in the union

CDSAT generalizes CDCL

- Transition rule Decide: ${}_{?}L$
  acceptable if $L \notin \Gamma$ and $\overline{L} \notin \Gamma$ (more later for first-order decisions)
- Transition rule Deduce adds justified assignment ${}_{J\vdash}L$
  with justification J if $J \vdash_k L$ for some $\mathcal{T}_k$
  $\mathrm{level}_\Gamma({}_{J\vdash}L) = \mathrm{level}_\Gamma(J)$ and $\mathrm{level}_\Gamma(J) = max\{\mathrm{level}_\Gamma(A) | A \in J\}$
  Deduce covers unit propagation: implied literal: ${}_{J\vdash}L$
  $J \vdash_{\mathrm{Bool}} L \qquad J = \{C \vee L, \neg C\}$
- Input assignments on Γ at level 0 as justified assignments with empty justification: ${}_{\emptyset\vdash}C$
- Trail not a stack: ${}_{J\vdash}L$ may be added after assignments of higher level if multiple modules share Γ (late propagation)

CDSAT generalizes CDCL

- Conflict: $J \subseteq \Gamma$, $J \vdash_k L$ for some $\mathcal{T}_k$, and $\overline{L} \in \Gamma$
  unsatisfiable assignment $E = J \cup \{\overline{L}\}$
- Conflict state: $\langle \Gamma; E \rangle$, $E \subseteq \Gamma$
- Transition rule Resolve explains E by replacing ${}_{J\vdash}L$ in E with J
- Given conflict $E = J \uplus H$ with cube $H = \{\overline{L}_1, \ldots, \overline{L}_k\}$
  transition rule LearnBackjump
  - Learns ${}_{J\vdash}C$ where $C = L_1 \vee \ldots \vee L_k$:
    J entails C since $J \uplus H$ is unsatisfiable
  - Backjumps to a level m such that
    $m < \text{level}_\Gamma(H)$ (quit conflict) and
    $m \geq \text{level}_\Gamma(J)$ so that ${}_{J\vdash}C$ can be added to Γ

First assertion clause heuristic in CDSAT

- Apply Resolve until conflict E contains only one literal $\overline{L}$ whose level m is max in E
- Generalize 1UIP: max in E not necessarily max in Γ
- Apply LearnBackjump to conflict $E = J \uplus H$ where $H = \{\overline{L}\} \uplus H'$ and $H' = \{\overline{L}_1, \ldots, \overline{L}_k\}$
- Learn ${}_{J\vdash} C$ where $C = L_1 \vee \ldots \vee L_k \vee L$ (first assertion clause)
- Backjump to level $n = \mathrm{level}_\Gamma(J \uplus H')$:
  $n < \mathrm{level}_\Gamma(H)$ as $\mathrm{level}_\Gamma(H) = \mathrm{level}_\Gamma(\overline{L})$ which is max in E
  $n \geq \mathrm{level}_\Gamma(J)$ as $J \uplus H'$ is superset of J
- Apply Deduce to add ${}_{\{C\}\uplus H'\vdash} L$ since $\{C\} \uplus H' \vdash_{\mathrm{Bool}} L$

LearnBackjump may follow other heuristics (e.g., learn and restart)

Example where CDSAT emulates CDCL

1. $S = \{\overline{A} \vee B,\ \overline{A} \vee C \vee E,\ \overline{B} \vee D,\ \overline{C} \vee \overline{D}, A \vee \overline{B} \vee E,\ B \vee \overline{C},\ F \vee \overline{E}\}$ subset of input
2. Decide adds ${}_{?}\overline{F}$ to trail Γ opening level n
3. Deduce adds ${}_{J\vdash}\overline{E}$ with $J = \{F \vee \overline{E}, {}_{?}\overline{F}\}$ to level n since $\{F \vee \overline{E}, {}_{?}\overline{F}\} \vdash_{\mathsf{Bool}} \overline{E}$
4. Two more Decide create levels $n+1$ and $n+2$
5. Another Decide adds ${}_{?}A$ opening level $n+3$
6. Deduce adds to level $n+3$
   ${}_{H\vdash}B$ with $H = \{\overline{A} \vee B, {}_{?}A\}$
   ${}_{I\vdash}C$ with $I = \{\overline{A} \vee C \vee E, {}_{J\vdash}\overline{E}, {}_{?}A\}$
   ${}_{K\vdash}D$ with $K = \{\overline{B} \vee D, {}_{H\vdash}B\}$

Example where CDSAT emulates CDCL

7. $\{\overline{C} \vee \overline{D},\ {}_{I\vdash}C\} \vdash_{\text{Bool}} \overline{D}$ but ${}_{K\vdash}D \in \Gamma$
   Conflict: $E_0 = \{\overline{C} \vee \overline{D},\ {}_{I\vdash}C,\ {}_{K\vdash}D\}$
   /* $\overline{C} \vee \overline{D}$ is conflict clause, not assertion clause */
8. E_0 contains literals ${}_{I\vdash}C$ and ${}_{K\vdash}D$ of max level $(n+3)$
   Resolve: $E_1 = \{\overline{C} \vee \overline{D},\ {}_{I\vdash}C,\ \overline{B} \vee D,\ {}_{H\vdash}B\}$
   /* $\overline{C} \vee \overline{D}$ and $\overline{B} \vee D$ yield $\overline{B} \vee \overline{C}$ (not assertion clause) */
9. E_1 contains literals ${}_{I\vdash}C$ and ${}_{H\vdash}B$ of max level $(n+3)$
   Resolve: $E_2 = \{\overline{C} \vee \overline{D},\ \overline{A} \vee C \vee E,\ {}_{J\vdash}\overline{E},\ {}_{?}A,\ \overline{B} \vee D,\ {}_{H\vdash}B\}$
   /* $\overline{B} \vee \overline{C}$ and $\overline{A} \vee C \vee E$ yield $\overline{B} \vee \overline{A} \vee E$ (not assertion clause) */
10. E_2 contains literals ${}_{?}A$ and ${}_{H\vdash}B$ of max level $(n+3)$
    Resolve: $E_3 = \{\overline{C} \vee \overline{D},\ \overline{A} \vee C \vee E,\ {}_{J\vdash}\overline{E},\ {}_{?}A,\ \overline{B} \vee D,\ \overline{A} \vee B\}$
    /* $\overline{B} \vee \overline{A} \vee E$ and $\overline{A} \vee B$ yield $\overline{A} \vee E$ (assertion clause) */

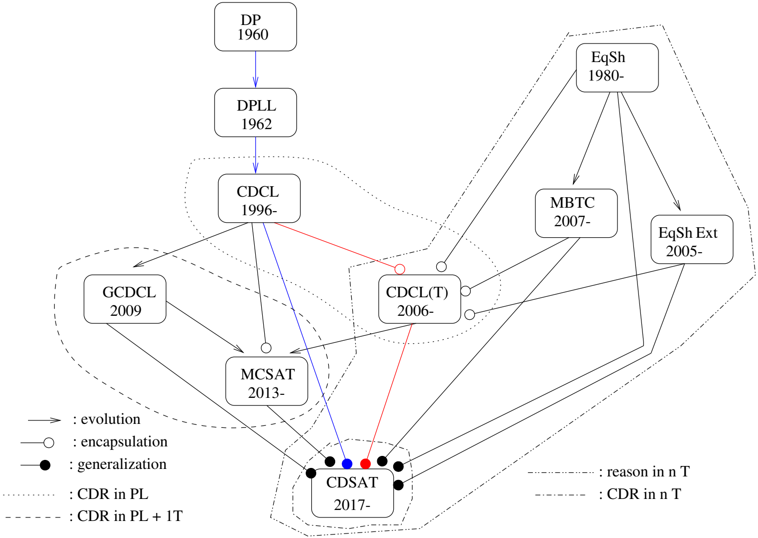
Example where CDSAT emulates CDCL

$E_3 = \{\overline{C} \vee \overline{D},\ \overline{A} \vee C \vee E,\ {}_{J\vdash}\overline{E},\ {}_{?}A,\ \overline{B} \vee D,\ \overline{A} \vee B\}$
${}_{?}A$ has level $n + 3$ (max), ${}_{J\vdash}\overline{E}$ has level n, and the rest has level 0

11. LearnBackjump jumps back to level n
    adds ${}_{G\vdash}(\overline{A} \vee E)$ with $G = \{\overline{C} \vee \overline{D},\ \overline{A} \vee C \vee E,\ \overline{B} \vee D,\ \overline{A} \vee B\}$
12. Deduce adds ${}_{M\vdash}\overline{A}$ with $M = \{{}_{G\vdash}(\overline{A} \vee E),\ {}_{J\vdash}\overline{E}\}$
    since $\{{}_{G\vdash}(\overline{A} \vee E), {}_{J\vdash}\overline{E}\} \vdash_{\mathrm{Bool}} \overline{A}$
13. Deduce adds ${}_{N\vdash}\overline{B}$ with $N = \{A \vee \overline{B} \vee E,\ {}_{M\vdash}\overline{A},\ {}_{J\vdash}\overline{E}\}$
14. Deduce adds ${}_{P\vdash}\overline{C}$ with $P = \{B \vee \overline{C},\ {}_{N\vdash}\overline{B}\}$

Γ contains $\{\overline{E},\ \overline{A},\ \overline{B},\ \overline{C}\}$ model of S

The big picture: from SAT to SMT



CDCL($\mathcal{T}$): from SAT to SMT

DPLL($\mathcal{T}$) later renamed CDCL($\mathcal{T}$) for $\mathcal{T}$ a single theory
[Nieuwenhuis, Oliveras, Tinelli: JACM 2006]

- CDCL + decision procedure for $\mathcal{T}$-satisfiability of set of $\mathcal{T}$-literals
- CDCL works on propositional abstraction: $\mathcal{T}$-atoms replaced by propositional variables
- Let $\{L_1, \ldots, L_n\} \subseteq \Gamma$ and $C = \overline{L}_1 \vee \ldots \vee \overline{L}_n$
  $\mathcal{T}$-sat procedure contributes only:
  - $\mathcal{T}$-conflict detection: if $\{L_1, \ldots, L_n\}$ is $\mathcal{T}$-unsat
    C is conflict clause
  - $\mathcal{T}$-propagation: if $\{L_1, \ldots, L_n\}$ $\mathcal{T}$-entails L
    add L to Γ with justification $C \vee L$
    L must be an input literal (i.e., not new)

CDCL($\mathcal{T}$): from SAT to SMT

- $\mathcal{T}$-sat procedure integrated as a black-box
- That only raises a flag if it detects an inconsistency in the propositional model that CDCL is building ignoring the theory:
  - $\mathcal{T}$-conflict: $\{L_1, \ldots, L_n\}$ is $\mathcal{T}$-unsat
    $\overline{L}_1 \vee \ldots \vee \overline{L}_n$ is $\mathcal{T}$-valid consequence of the input
  - $\mathcal{T}$-propagation: $\{L_1, \ldots, L_n, \overline{L}\}$ is $\mathcal{T}$-unsat
    $\overline{L}_1 \vee \ldots \vee \overline{L}_n \vee L$ is $\mathcal{T}$-valid consequence of the input

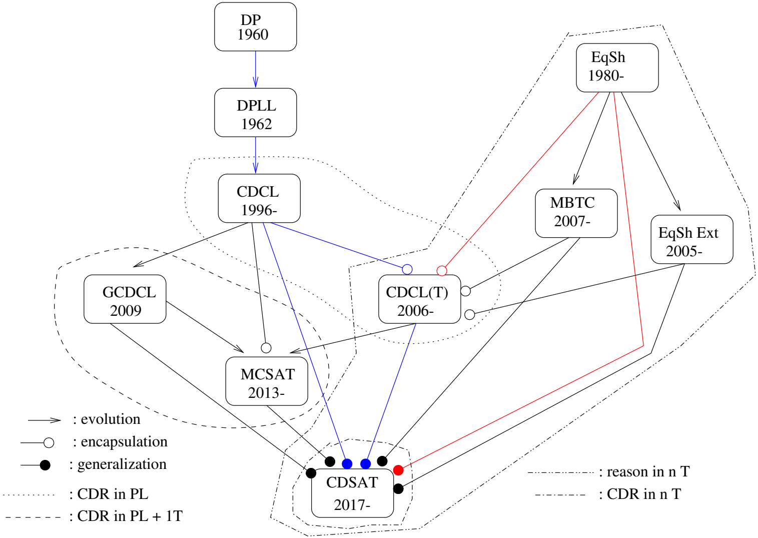
  Never deduce anything that excludes a $\mathcal{T}$-model but is not a $\mathcal{T}$-valid consequence of the input
- Model search, trail, conflict explanation, conflict-driven reasoning remain propositional

CDSAT generalizes CDCL($\mathcal{T}$)

- Consider a theory union whose members are Bool and $\mathcal{T}$
- Theory modules:
  - Bool-module
  - Black-box $\mathcal{T}$-module:
    - Only one inference rule: $L_1, \ldots, L_m \vdash \bot$
    - That invokes the $\mathcal{T}$-procedure to detect $\mathcal{T}$-unsat of a set of literals

CDSAT can use a black-box $\mathcal{T}$-module
whenever a theory $\mathcal{T}$ does not have a conflict-driven procedure

The big picture: theory combination



Classical approach to theory combination: equality sharing

Equality sharing aka Nelson-Oppen method
[Nelson, Oppen: ACM TOPLAS 1979]

- $\mathcal{T} = \bigcup_{k=1}^{n} \mathcal{T}_k$: disjoint theories (share $\simeq$ and sorts)
- Decision procedure for $\mathcal{T}_k$-satisfiability of set of $\mathcal{T}_k$-literals
- Stably infinite: $\mathcal{T}_k$-model with infinite cardinality
- Get decision procedure for $\mathcal{T}$-satisfiability of set of $\mathcal{T}$-literals
- Combination of decision procedures as black-boxes
- By disjointness, agreement is needed on:
  - Cardinalities of shared sorts: by stable infiniteness
  - Equalities between shared terms: needs work

Equality sharing: separation

- Input set S: $\mathcal{T}$-literals mix symbols from the $\mathcal{T}_k$'s signatures
- Separate S into sets S_k of $\mathcal{T}_k$-literals
  sharing only $\simeq$ and variables

Example: S contains $f(2, y) \simeq f(x, y)$

- EUF ($f \in \Sigma_{\mathrm{EUF}}$) and LIA ($2 \in \Sigma_{\mathrm{LIA}}$)
- Shared sort: Z; $\quad \simeq$ is $\simeq_{\mathrm{Z}}$; $\quad f : \mathrm{Z} \times \mathrm{Z} \rightarrow \mathrm{Z}$
- EUF: 2 is a variable
- LIA: $f(2, y)$ and $f(x, y)$ are variables
- $S_{\mathrm{EUF}} = \{w_1 \simeq f(w_2, y),\ w_3 \simeq f(x, y),\ w_1 \simeq w_3\}$
- $S_{\mathrm{LIA}} = \{w_2 \simeq 2,\ w_1 \simeq w_3\}$
- Shared variables: $\mathcal{V}_{\mathrm{sh}}(S) = \{w_1,\ w_2,\ w_3\}$

How CDSAT handles separation

- Input set S: $\mathcal{T}$-literals mix symbols from the $\mathcal{T}_k$'s signatures
- Each $\mathcal{T}_k$ treats as a variable a term whose top symbol is foreign

Example: S contains $f(2,y) \simeq f(x,y)$
(i.e., $(f(2,y) \simeq f(x,y)) \leftarrow \mathsf{true}$)

- EUF ($f \in \Sigma_{\mathrm{EUF}}$) and LIA ($2 \in \Sigma_{\mathrm{LIA}}$)
- Shared sort: Z; $\quad \simeq$ is $\simeq_{\mathrm{Z}}$; $\quad f\colon \mathrm{Z} \times \mathrm{Z} \rightarrow \mathrm{Z}$
- EUF: 2 is foreign hence a variable
- LIA: f is foreign hence $f(2,y)$ and $f(x,y)$ are variables
- Shared terms:
  $\mathcal{V}_{\mathrm{sh}}(S) = \{f(2,y) \simeq f(x,y),\ f(2,y),\ 2,\ f(x,y)\}$

Equality sharing: the reduction

- Reduce the $\mathcal{T}$-sat problem to $\mathcal{T}_k$-sat problems
- S is $\mathcal{T}$-sat iff $\bigcup_{k=1}^{n} S_k$ is $\mathcal{T}$-sat
- Arrangement α: represents a partition of $\mathcal{V}_{\mathrm{sh}}(S)$
- α: conjunction that contains
  - $u \simeq v$ if u and v in the same class of the partition
  - $u \not\simeq v$ otherwise
- Combination theorem:
  $\bigcup_{k=1}^{n} S_k$ is $\mathcal{T}$-sat iff $\exists\ \alpha$ s.t. $S_k \wedge \alpha$ is $\mathcal{T}_k$-sat $(1 \leq k \leq n)$

Equality sharing: build arrangement (convex theories)

- $\mathcal{E}_0 = \emptyset$
- $\mathcal{E}_i = \mathcal{E}_{i-1} \cup \{u \simeq v\}$ if a $\mathcal{T}_k$-sat procedure deduces $u \simeq v$ from $S_k \cup \mathcal{E}_{i-1}$
- If a $\mathcal{T}_k$-sat procedure deduces $\perp$ from $S_k \cup \mathcal{E}_i$ for some i: return unsat (S is $\mathcal{T}$-unsat)
- Otherwise, let $\alpha = \mathcal{E}_q$ such that $\mathcal{E}_q = \mathcal{E}_{q-1}$ (no more equalities) and return sat (S is $\mathcal{T}$-sat)

Complete for convex theories:
$\mathcal{T}_k$ is convex if
$\mathcal{T}_k \models H \supset \bigvee_{i=1}^{n} u_i \simeq v_i$ implies $\exists\, j,\ 1 \leq j \leq n,\ \mathcal{T}_k \models H \supset u_j \simeq v_j$
H: a conjunction of $\mathcal{T}_k$-literals

Equality sharing: build arrangement (non-convex theories)

- $\mathcal{T}_k$ not convex: $\mathcal{T}_k$-procedure deduces $\bigvee_{j=1}^{m} u_j \simeq v_j$
- $\mathcal{T}$-procedure calls itself recursively on each subproblem obtained by adding $u_j \simeq v_j$ to current $\mathcal{E}_i$
- In practice: CDCL($\mathcal{T}$) where $\mathcal{T}$-procedure is equality sharing combination [Barrett, Nieuwenhuis, Oliveras, Tinelli: LPAR 2006] [Krstić, Amit Goel: FroCoS 2007]
  - $\mathcal{T}$-procedure sends (propositional abstraction of) $\bigvee_{j=1}^{m} u_j \simeq v_j$ to CDCL
  - Reasoning about disjunction is entrusted to CDCL
  - Case $u_j \simeq v_j$ is considered when CDCL puts it on the trail
  - Sole new (i.e., non-input) literals in CDCL($\mathcal{T}$): (propositional abstractions of) equalities between shared variables

Equality sharing is not conflict-driven

- Combining theories by combining procedures
- $\mathcal{T}_k$-procedures combined as black-boxes
- Generation of (disjunctions of) equalities resembles saturation (can be emulated by superposition)
- In CDCL($\mathcal{T}$) where $\mathcal{T}$-procedure is equality sharing combination, model search, trail, conflict explanation, conflict-driven reasoning remain propositional

In order to see how CDSAT emulates Equality Sharing, let's learn more about theory modules in CDSAT

CDSAT modules

Theory modules $\mathcal{I}_1, \ldots, \mathcal{I}_n$ for theories $\mathcal{T}_1, \ldots, \mathcal{T}_n$

- Theory module $\mathcal{I}_k$ for theory $\mathcal{T}_k$ is a set of inference rules $J \vdash_k L$ where
  - J is a $\mathcal{T}_k$-assignment: may contain first-order assignments
  - L is a singleton Boolean assignment
  - If a first-order assignment to x follows from the trail it can be added as a decision (forced decision)
- Local basis: $\text{basis}_k(X)$ contains all terms that $\mathcal{I}_k$ can generate from set of terms X

CDSAT modules: equality inferences

All CDSAT theory modules include equality inferences:

- Reflexivity: $\vdash t \simeq t$
- Symmetry: $t \simeq s \vdash s \simeq t$
- Transitivity: $t \simeq s,\ s \simeq u \vdash t \simeq u$
- Same value: $t \leftarrow \mathfrak{c},\ s \leftarrow \mathfrak{c} \vdash t \simeq s$
- Different values: $t \leftarrow \mathfrak{c},\ s \leftarrow \mathfrak{q} \vdash t \not\simeq s$

With first-order assignments, two ways to make $t \simeq s$ true:
$(t \simeq s) \leftarrow \mathrm{true}$ and $t \leftarrow \mathfrak{c},\ s \leftarrow \mathfrak{c}$

CDSAT generalizes equality sharing

- Each $\mathcal{T}_k$ module can place its inferences $J \vdash_k L$ as justified assignments ${}_{J\vdash}L$ on the shared trail by Deduce transitions (Deduce covers $\mathcal{T}_k$-propagation)
  - Equality inferences: transitivity steps and equalities from first-order assignments contribute to build an arrangement
  - Theory specific inference rules can deduce (disjunctions of) equalities
- The $\mathcal{T}_k$ modules cooperate to build an arrangement publicly on the shared trail
- Disjunctions are handled by the Bool-module by decision and unit propagation (as in CDCL)

CDSAT module for equality with uninterpreted functions

- $\Sigma_{\mathsf{EUF}} = \langle S, F\rangle$ prop $\in S$ $\simeq_s \in F$ for all sorts $s \in S$
- EUF$^+$ may be trivial or add countably many values for each $s \in S \setminus \{\text{prop}\}$ used as labels of congruence classes, e.g.:
  $t_1 \leftarrow \mathfrak{c},\ t_2 \leftarrow \mathfrak{c},\ t_3 \leftarrow \mathfrak{c}_3,\ t_4 \leftarrow \mathfrak{c}_4,\ t_5 \leftarrow \mathfrak{c}_5$
  shorter than
  $t_1 \simeq t_2,\ t_1 \not\simeq t_3,\ t_1 \not\simeq t_4,\ t_1 \not\simeq t_5,\ t_3 \not\simeq t_4,\ t_3 \not\simeq t_5,\ t_4 \not\simeq t_5$
- Congruence:
  - $(t_i \simeq u_i)_{i=1\dots m}, (f(t_1,\dots,t_m) \not\simeq f(u_1,\dots,u_m)) \vdash_{\mathsf{EUF}} \bot$
  - $(t_i \simeq u_i)_{i=1\dots m} \vdash_{\mathsf{EUF}} f(t_1,\dots,t_m) \simeq f(u_1,\dots,u_m)$
  - $(t_i \simeq u_i)_{i=1\dots m, i\neq j}, f(t_1,\dots,t_m) \not\simeq f(u_1,\dots,u_m) \vdash_{\mathsf{EUF}} t_j \not\simeq u_j$
- $\text{basis}_{\mathsf{EUF}}(X)$: all subterms of terms in X and all equalities between them

Example where CDSAT emulates equality sharing

1. $\{x \leq y,\ y \leq (x + g(x)),\ P(h(x) - h(y)),\ \neg P(0),\ g(x) \simeq 0\}$
   Theory union: LIA $\cup$ EUF
2. $S = \{x \leq y, y \leq (x + g(x)), f(h(x) - h(y)) \simeq \bullet, f(0) \not\simeq \bullet, g(x) \simeq 0\}$
   $\mathcal{V}_{sh}(S) = \{x,\ y,\ g(x),\ h(x),\ h(y),\ h(x) - h(y),\ 0\}$
3. LIA-module: $\{y \leq x + g(x),\ g(x) \simeq 0\} \vdash_{\mathsf{LIA}} y \leq x$
   Deduce: ${}_{J}{\vdash}(y \leq x)$ (level 0)
   with $J = \{y \leq x + g(x),\ g(x) \simeq 0\}$
   /* step hidden in black-box LIA-procedure in equality sharing */
4. LIA-module: $\{x \leq y,\ {}_{J}{\vdash}(y \leq x)\} \vdash_{\mathsf{LIA}} x \simeq y$
   Deduce: ${}_{H}{\vdash}(x \simeq y)$ (level 0)
   with $H = \{x \leq y,\ {}_{J}{\vdash}(y \leq x)\}$

Example where CDSAT emulates equality sharing

5. EUF-module: ${}_{H\vdash}(x \simeq y) \vdash_{\mathsf{EUF}} h(x) \simeq h(y)$
   Deduce: ${}_{I\vdash}(h(x) \simeq h(y))$ (level 0)
   with $I = \{{}_{H\vdash}(x \simeq y)\}$
6. LIA-module: ${}_{I\vdash}(h(x) \simeq h(y)) \vdash_{\mathsf{LIA}} h(x) - h(y) \simeq 0$
   Deduce: ${}_{K\vdash}(h(x) - h(y) \simeq 0)$ (level 0)
   with $K = \{{}_{I\vdash}(h(x) \simeq h(y))\}$
7. EUF-module:
   $\{f(h(x) - h(y)) \simeq \bullet,\ {}_{K\vdash}(h(x) - h(y) \simeq 0)\} \vdash_{\mathsf{EUF}} f(0) \simeq \bullet$
   but the trail contains $f(0) \not\simeq \bullet$
   EUF-conflict:
   $E = \{f(h(x) - h(y)) \simeq \bullet,\ {}_{K\vdash}(h(x) - h(y) \simeq 0),\ f(0) \not\simeq \bullet\}$
   (level 0)
   Fail returns unsat (nowhere to backjump to)

CDSAT can emulate equality sharing

- Each $\mathcal{T}_k$ module can also place decisions on the shared trail by Decide transitions
- A $\mathcal{T}_k$-inference $J \vdash_k L$ from $J \subseteq \Gamma$ leads to $\mathcal{T}_k$-conflict $E = J \cup \{\overline{L}\}$ if $\overline{L} \in \Gamma$
- Solved by LearnBackjump

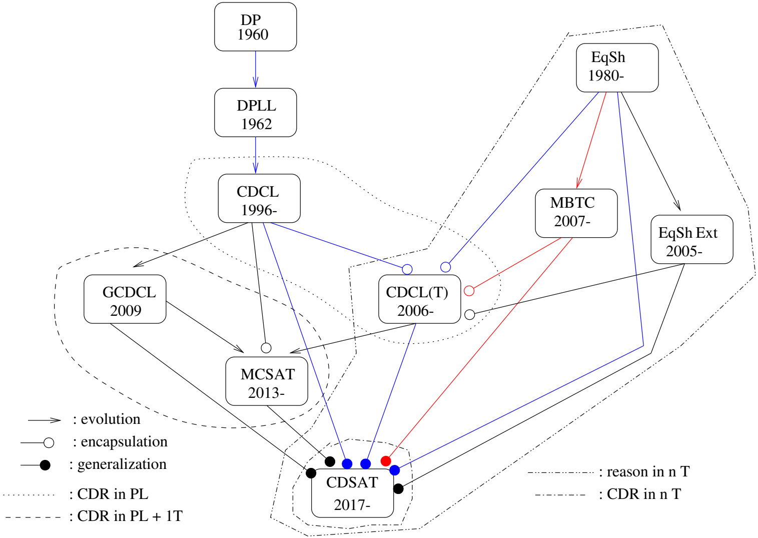
Example where CDSAT emulates equality sharing: variant

1. $\{x \leq y,\ y \leq (x + g(x)),\ P(h(x) - h(y)),\ \neg P(0),\ g(x) \simeq 0\}$
   theories: LIA $\cup$ EUF
2. $S{=}\{x{\leq}y, y{\leq}(x{+}g(x)), f(h(x){-}h(y)) \simeq \bullet, f(0) \not\simeq \bullet, g(x) \simeq 0\}$
   $\mathcal{V}_{\mathrm{sh}}(S) = \{x,\ y,\ g(x),\ h(x),\ h(y),\ h(x) - h(y),\ 0\}$
3. EUF-module: Decide adds $_{?}(x \not\simeq y)$ (level 1)
4. LIA-module: $\{y \leq x + g(x),\ g(x) \simeq 0\} \vdash_{\mathrm{LIA}} y \leq x$
   Deduce: $_{J\vdash}(y \leq x)$ (level 0)
   with $J = \{y \leq x + g(x),\ g(x) \simeq 0\}$ /* late propagation */
5. LIA-module: $\{x \leq y,\ _{J\vdash}(y \leq x)\} \vdash_{\mathrm{LIA}} x \simeq y$
   but the trail contains $_{?}(x \not\simeq y)$
   LIA-conflict: $E_0 = \{_{?}(x \not\simeq y),\ x \leq y,\ _{J\vdash}(y \leq x)\}$

Example where CDSAT emulates equality sharing: variant

6. LIA-conflict: $E_0 = \{_?(x \not\simeq y),\ x \leq y,\ _{J\vdash}(y \leq x)\}$
   $_?(x \not\simeq y)$ has level 1, the rest has level 0
7. LearnBackjump: back to level 0 adding $_{H\vdash}(x \simeq y)$
   $H = \{x \leq y,\ _{J\vdash}(y \leq x)\}$
   the derivation continues as before

The big picture: more theory combination



Model-based theory combination (MBTC)

[de Moura, Bjørner: SMT 2007]

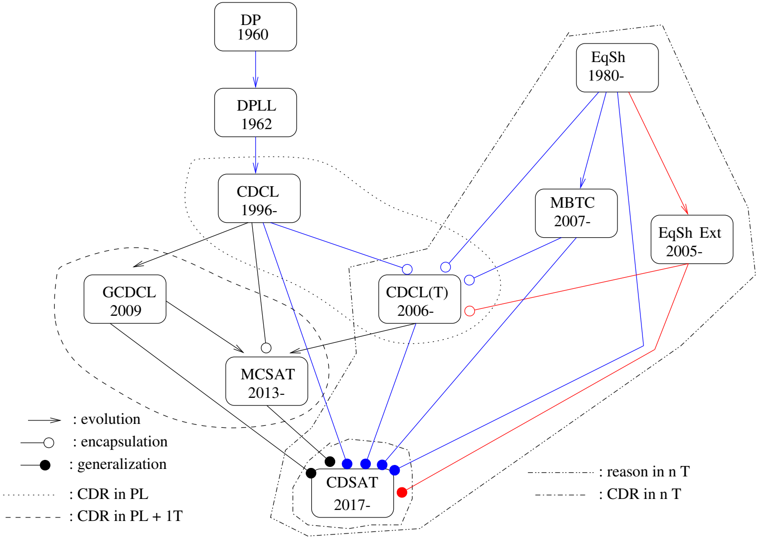
- Variant of equality sharing in CDCL($\mathcal{T}$)
- Assume $\mathcal{T}_k$-sat procedure builds candidate model $\mathcal{M}_k$ (e.g., linear arithmetic)
- Share $u \simeq v$ if true in $\mathcal{M}_k$ not necessarily $\mathcal{T}_k$-entailed by $S_k \cup \mathcal{E}_i$ (u and v $\mathcal{T}_k$-terms occurring in S_k)
- (Propositional abstraction of) $u \simeq v$ posted on trail as decision
- If $\mathcal{T}_k$-conflict ensues, undo, and update $\mathcal{M}_k$
- Useful to accelerate reaching sat

$\mathcal{M}_k$ and conflict-driven updates remain inside black-box procedure

CDSAT generalizes MBTC

- All theory modules cooperate as peers to build a model for $\mathcal{T} = \bigcup_{k=1}^{n} \mathcal{T}_k$ on the shared trail
- A theory module $\mathcal{I}_k$ can build a partial $\mathcal{T}_k$-model $\mathcal{M}_k$ publicly on the shared trail
- $\mathcal{I}_k$ can deduce an equality $u \simeq v$ that follows from assignments in $\mathcal{M}_k$: CDSAT modules deduce from first-order assignments
- If a conflict ensues, $u \simeq v$ and the first-order decisions from which it depends will be undone, and $\mathcal{M}_k$ will be amended
- MBTC does it with a decision, because in CDCL($\mathcal{T}$) only $\mathcal{T}$-valid consequences of the input can be deduced

The big picture: more theory combination



Extensions of equality sharing

[Tinelli, Zarba: JAR 2005] [Fontaine: FroCoS 2009]
[Jovanović, Barrett: LPAR 2010] [Toledo, Przybocki, Zohar: CADE 2025]

- Variants of equality sharing in CDCL($\mathcal{T}$)
- Equality sharing requires the theories to be stably infinite
- Variants allow $\mathcal{T}_1$ not stably infinite, if $\mathcal{T}_2$ satisfies stronger cardinality requirements
- Still combining theories by combining procedures
- Procedures combined as black-boxes
- Completeness approach as in equality sharing: hypotheses on theories + combination theorem

CDSAT does not require stable infiniteness

CDSAT and agreement on cardinalities of sorts

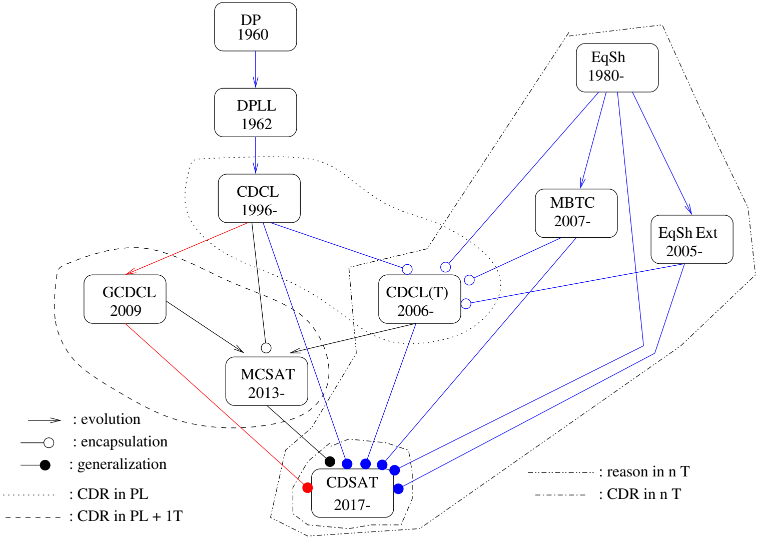
- CDSAT requires that there exists leading theory, say $\mathcal{T}_1$, that
  - Has all sorts in the theory union
  - Has all cardinality constraints aggregated and enforced by $\mathcal{T}_1$-module inferences
- Every $\mathcal{T}_k$ $(k \neq 1)$ has to agree with $\mathcal{T}_1$ on what's shared: any two $\mathcal{T}_k$ and $\mathcal{T}_j$ $(k \neq j)$ agree
- Agreement guaranteed by theory modules completeness requirements:
  - $\mathcal{T}_1$-module complete
  - $\mathcal{T}_k$-module $(k \neq 1)$ leading-theory-complete
- CDSAT approach to completeness differs from that of (variants of) equality sharing

Examples

1. All theories stably infinite: $\mathcal{T}_1$ is fictional $\mathcal{T}_{\mathbb{N}}$ that interprets all sorts (except prop) as having the cardinality of $\mathbb{N}$
2. At-most-m cardinality constraint on sort s:
   $\forall x_0, \ldots, \forall x_m.\ \bigvee_{0 \leq i \neq k \leq m} x_i \simeq_s x_k$
   $x_0, \ldots, x_m$: $m+1$ distinct variables of sort s

   Inference rule in the $\mathcal{T}_1$-module:
   $\bigwedge_{0 \leq i \neq k \leq m} u_i \not\simeq_s u_k \vdash_{\mathcal{T}_1} \bot$
   $u_0, \ldots u_m$: any $m+1$ distinct terms of sort s
3. Aggregation: if $\mathcal{T}_2$ says at-most-m and $\mathcal{T}_2$ says at-most-p, $\mathcal{T}_1$ says at-most-$min(m,p)$

The big picture: conflict-driven theory reasoning



Conflict-driven satisfiability procedures in arithmetic

Generalize the CDCL pattern:

- Candidate model: theory model (e.g., LRA, LIA, NRA)
- Assignment: also to first-order terms
  (e.g., $x \leftarrow 3$, $x < y \leftarrow \text{true}$, $z \leftarrow y + 3$)
- Propagation: also evaluation of arithmetic expressions
  (e.g., $y \leftarrow 0 \vdash_{\mathsf{LRA}} (y > 2) \leftarrow \text{false}$)
- Explanation: also theory-conflicts by theory inferences
- Learn lemmas that may contain new (non-input) atoms and may exclude first-order assignments
- Expensive theory inferences only on demand to respond to conflicts

Outline of GCDCL procedure for generic single theory $\mathcal{T}$

[McMillan, Kuehlmann, Sagiv: CAV 2009]

- Embed reasoning about disjunction into theory reasoning by generalizing to $\mathcal{T}$-clauses a theory reasoning inference rule for $\mathcal{T}$-literals
- Apply the generalized rule only to explain conflicts
- Devise restrictions to ensure termination

Achieved in GCDCL: linear rational arithmetic (LRA)

Linear rational arithmetic (LRA)

- Input: set S of LRA-clauses
- LRA-term: rational constant c, sum $c_1 \cdot x_1 + \ldots + c_n \cdot x_n$
- LRA-clause: disjunction of $t_1 \lessdot t_2$ literals, $\lessdot \in \{<, \leq\}$
- $\overline{(t_1 < t_2)}$ and $\overline{(t_1 \leq t_2)}$ replaced by $t_2 \leq t_1$ and $t_2 < t_1$
- $t_1 \simeq t_2$ rewritten as $t_1 \leq t_2$ and $t_2 \leq t_1$
- Variable x with positive coefficient:
  rearrange literal into upper bound $x \lessdot t$
- Variable x with negative coefficient:
  rearrange literal into lower bound $t \lessdot x$

Linear rational arithmetic (LRA)

- Fourier-Motzkin (FM) resolution:
  $\{t_1 \lessdot_1 x,\ x \lessdot_2 t_2\} \vdash_{\mathrm{LRA}} t_1 \lessdot_3 t_2$
  $\lessdot_1, \lessdot_2, \lessdot_3 \in \{<, \leq\}$
  $\lessdot_3$ is $<$ if either $\lessdot_1$ or $\lessdot_2$ is $<$ and $\leq$ otherwise
  - Transitive closure: $\{x < -y,\ -y < -2\} \vdash_{\mathrm{LRA}} x < -2$
  - Linear combination of constraints:
    $\{x + y < 0,\ -y + 2 < 0\} \vdash_{\mathrm{LRA}} x + 2 < 0$
- Fourier-Motzkin algorithm:
  termination guaranteed
  (elim one variable at each round, finitely many variables)
  but generates a doubly exponential number of constraints

[Lassez, Maher: JAR 1992]

Generalized CDCL (GCDCL) for LRA

[McMillan, Kuehlmann, Sagiv: CAV 2009]

- Generalize FM-resolution to LRA-clauses: shadow rule e.g.:
  $\{(b < d) \vee (c < d),\ d < a\} \vdash_{\mathsf{LRA}} (b < a) \vee (c < a)$
- Generates new (non-input) atoms
- Applied only to explain LRA-conflicts
  generating lemmas excluding LRA-assignments
- Add restrictions to recover termination:
  assume fixed total ordering $\prec_{\mathsf{LRA}}$ on rational variables
  apply inference only if the variable resolved upon is
  $\prec_{\mathsf{LRA}}$-maximum in both premises

Independently:

[Korovin, Tsiskaridze, Voronkov: CP 2009] [Cotton: FORMATS 2010]

CDSAT module for linear rational arithmetic (LRA)

- Signature Σ_{LRA}:
  - Sorts: $S = \{\mathsf{prop}, \mathsf{Q}\}$
  - Symbols: $\simeq_s$ for all $s \in S$
    $1, +, <, \leq, q\cdot$ for all rational numbers $q \in \mathbb{Q}$
- Theory extension LRA^+ adds constants $\tilde{q}$ for all $q \in \mathbb{Q}$
- Inference rules:
  - Evaluation: $(t_1 \leftarrow \tilde{q}_1, \ldots, t_m \leftarrow \tilde{q}_m) \vdash_{\mathsf{LRA}} l \leftarrow \mathfrak{b}$
  - Disequality elimination:
    $t_1 \leq x,\ x \leq t_2,\ t_1 \simeq_{\mathsf{Q}} t_0,\ t_2 \simeq_{\mathsf{Q}} t_0,\ x \not\simeq_{\mathsf{Q}} t_0 \vdash_{\mathsf{LRA}} \perp$
    detects LRA-conflict: no value for variable x

CDSAT module for linear rational arithmetic (LRA)

- FM-resolution: $\{t_1 \lessdot_1 x,\ x \lessdot_2 t_2\} \vdash_{\mathsf{LRA}} t_1 \lessdot_3 t_2$
  $\lessdot_1, \lessdot_2, \lessdot_3 \in \{<, \leq\}$
  $\lessdot_3$ is $<$ if either $\lessdot_1$ or $\lessdot_2$ is $<$ and $\leq$ otherwise
- $\mathrm{basis}_{\mathsf{LRA}}(X)$: subterms, equalities, disequalities restricting FM-resolution to resolve on the $\prec_{\mathsf{LRA}}$-maximum variable
- Detection of empty solution space:
  $\{y_1 \leftarrow \tilde{q}_1, \ldots, y_m \leftarrow \tilde{q}_m\} \uplus E \vdash_{\mathsf{LRA}} \bot$
  for all x in E, $x \prec_{\mathsf{LRA}} y_i$ or $x = y_i$ for some i $(1 \leq i \leq m)$
- Alternatively and in practice: sensible search plan that selects rational variables for decision in $\prec_{\mathsf{LRA}}$-increasing order

More on CDSAT

For CDSAT at work on conflict-driven theory reasoning, we need:

- Acceptability of first-order decisions
- Transition rule Deduce beyond unit propagation and deduction of equalities between shared terms
- Transition rule to solve conflicts due to first-order decisions: UndoClear

Let's also have a more formal look at the CDSAT trail

CDSAT trail: a sequence of assignments

- Each assignment is a decision ${}_{?}A$ or a justified assignment ${}_{H\vdash}A$
- Decision: either Boolean or first-order; opens the next level
- Justification of A: set H of assignments that appear before A
  - Due to an inference $H \vdash_k A$
  - Due to conflict-solving transitions
  - Boolean or input first-order assignment
  - Input assignment ($H = \emptyset$)
- Level of A: max among those of the elements of H
- A justified assignment of level 5 may appear after a decision of level 6: late propagation; a trail is not a stack

Acceptability of a decision

- Boolean decision ${}_?L$: it suffices $L \notin \Gamma$ and $\overline{L} \notin \Gamma$
- First-order decision ${}_?(u \leftarrow \mathfrak{c})$
  where $\mathfrak{c}$ is a $\mathcal{T}_k$-value:
  - Trail Γ does not assign a $\mathcal{T}_k$-value to term u
  - $u \leftarrow \mathfrak{c}$ does not trigger a $\mathcal{T}_k$-inference $J \cup \{u \leftarrow \mathfrak{c}\} \vdash_k \overline{L}$
    with $J \subseteq \Gamma$ and $L \in \Gamma$
  - Excluding a first-order decision that triggers an immediate conflict from which nothing can be learned

CDSAT transition rule Deduce

- Propagation:
  - Boolean propagation: e.g., unit propagation
  - $\mathcal{T}_k$-propagation: e.g., propagation of equalities when emulating equality sharing
- $\mathcal{T}_k$-inferences that explain a $\mathcal{T}_k$-conflict
  generating lemmas possibly excluding $\mathcal{T}_k$-assignments
  until the $\mathcal{T}_k$-conflict can be detected
  as a Boolean conflict on the trail:
  $J \vdash_k L$ and $\overline{L} \in \Gamma$
  unsatisfiable assignment $E = J \cup \{\overline{L}\}$

CDSAT transition rule UndoClear

- The assignment of max level in the conflict is a first-order decision
- A first-order assignment does not have a complement that can be learned
- UndoClear incorporates backtracking from the level of the bad decision to the previous one
- The state has changed due to a late propagation
- UndoClear fires after a late propagation:
  bad decision was acceptable prior to the late propagation;
  causes a conflict afterwards

Example with UndoClear

$\{l_0 : 2x + y \simeq 1,\ l_1 : 2x + 2y \simeq 1\}$ subset of the input (level 0)

1. Decide: ${}_?(x \leftarrow 0)$ (level 1) /* acceptable */
2. Deduce: ${}_{J\vdash}(y \simeq 0)$ with $J = \{2x + y \simeq 1,\ 2x + 2y \simeq 1\}$ (level 0)
   FM-resolution: $\{2x + y \simeq 1,\ 2x + 2y \simeq 1\} \vdash_{\mathrm{LRA}} y \simeq 0\ (l_1 - l_0)$
   /* late propagation */
3. $\{{}_?(x \leftarrow 0),\ {}_{J\vdash}(y \simeq 0)\} \vdash_{\mathrm{LRA}} 2x + y \not\simeq 1$ detects
   LRA-conflict $E = \{{}_?(x \leftarrow 0),\ {}_{J\vdash}(y \simeq 0),\ 2x + y \simeq 1\}$
   UndoClear: undo ${}_?(x \leftarrow 0)$ (max level in E) back to level 0
4. Decide: ${}_?(x \leftarrow 1/2)$ (level 1)
   /* forced decision: only acceptable value for x */

Example of non-termination of FM-resolution

Infinite sequence of FM-resolutions alternating on distinct variables:

$$
\begin{array}{lcll}
l_0 : & -2{\cdot}x - y < 0 & & \\
l_1 : & x + y < 0 & & \\
l_2 : & x < -1 & & \\
l_3 : & -y < -2 & (l_0 + 2l_2) & \mathit{elim}\ x \\
l_4 : & x < -2 & (l_1 + l_3) & \mathit{elim}\ y \\
l_5 : & -y < -4 & (l_0 + 2l_4) & \mathit{elim}\ x \\
l_6 : & x < -4 & (l_1 + l_5) & \mathit{elim}\ y \\
l_7 : & -y < -8 & (l_0 + 2l_6) & \mathit{elim}\ x \\
\ldots & \ldots & \ldots & \ldots
\end{array}
$$

It may arise even if FM-resolution is applied
only to respond to LRA-conflicts

Example where CDSAT emulates GCDCL

$l_0: -2{\cdot}x - y < 0,\ l_1: x + y < 0,\ l_2: x < -1$ (level 0)

1. Decide: ${}_?(y{\leftarrow}0)$ (level 1) /* acceptable */
   LRA-conflict: $\{-2{\cdot}x - y < 0,\ x < -1,\ y{\leftarrow}0\}$
2. Explained by $l_0 + 2l_2: \{-y < 2{\cdot}x,\ 2{\cdot}x < -2\} \vdash_{\mathsf{LRA}} -y < -2$
   Deduce: $l_3: -y < -2$ (level 0) /* late propagation */
3. $y{\leftarrow}0 \vdash_{\mathsf{LRA}} \overline{-y < -2}$ detects LRA-conflict $\{y{\leftarrow}0,\ -y < -2\}$
   UndoClear: undo ${}_?(y{\leftarrow}0)$ and back to level 0
4. Decide: ${}_?(x{\leftarrow}-2)$ (level 1) /* acceptable */
   LRA-conflict: $\{x + y < 0,\ -y < -2,\ x{\leftarrow}-2\}$
5. Explained by $l_1 + l_3: \{x < -y,\ -y < -2\} \vdash_{\mathsf{LRA}} x < -2$
   Deduce: $l_4: x < -2$ (level 0) /* late propagation */

Example where CDSAT emulates GCDCL

6. $x \leftarrow -2 \vdash_{\mathsf{LRA}} \overline{x < -2}$ detects LRA-conflict $\{x \leftarrow -2,\ x < -2\}$
   UndoClear: undo $_{?}(x \leftarrow -2)$ and back to level 0
7. Decide: $_{?}(y \leftarrow 3)$ (level 1) /* acceptable */
   LRA-conflict: $\{-2 \cdot x - y < 0,\ x < -2,\ y \leftarrow 3\}$
8. Explained by $l_0 + 2l_4$: $\{-y < 2 \cdot x,\ 2 \cdot x < -4\} \vdash_{\mathsf{LRA}} -y < -4$
   Deduce: l_5: $-y < -4$ (level 0) /* late propagation */
9. $y \leftarrow 3 \vdash_{\mathsf{LRA}} \overline{-y < -4}$ detects LRA-conflict $\{y \leftarrow 3,\ -y < -4\}$
   UndoClear: undo $_{?}(y \leftarrow 3)$ and back to level 0
10. Decide: $_{?}(x \leftarrow -3)$ (level 1) /* acceptable */
    LRA-conflict: $\{x + y < 0,\ -y < -4,\ x \leftarrow -3\}$

Example where CDSAT emulates GCDCL

11. Explained by $l_1 + l_5$: $\{x < -y,\ -y < -4\} \vdash_{\mathsf{LRA}} x < -4$
    Deduce: $l_6 : x < -4$ (level 0) /* late propagation */
12. $x \leftarrow -3 \vdash_{\mathsf{LRA}} \overline{x < -4}$ detects LRA-conflict $\{x \leftarrow -3,\ x < -4\}$
    UndoClear: undo $_?(x \leftarrow -3)$ and back to level 0
13. Decide: $_?(y \leftarrow 5)$ (level 1) /* acceptable */
    LRA-conflict: $\{-2 \cdot x - y < 0,\ x < -4,\ y \leftarrow 5\}$
14. Explained by $l_0 + 2l_6$: $\{-y < 2 \cdot x,\ 2 \cdot x < -8\} \vdash_{\mathsf{LRA}} -y < -8$
    Deduce: $l_7 : -y < -8$ (level 0) /* late propagation */
15. $y \leftarrow 5 \vdash_{\mathsf{LRA}} \overline{-y < -8}$ detects LRA-conflict $\{y \leftarrow 5,\ -y < -8\}$
    UndoClear: undo $_?(y \leftarrow 5)$ and back to level 0

$\dots \quad \dots \quad \dots$

Example where CDSAT emulates GCDCL

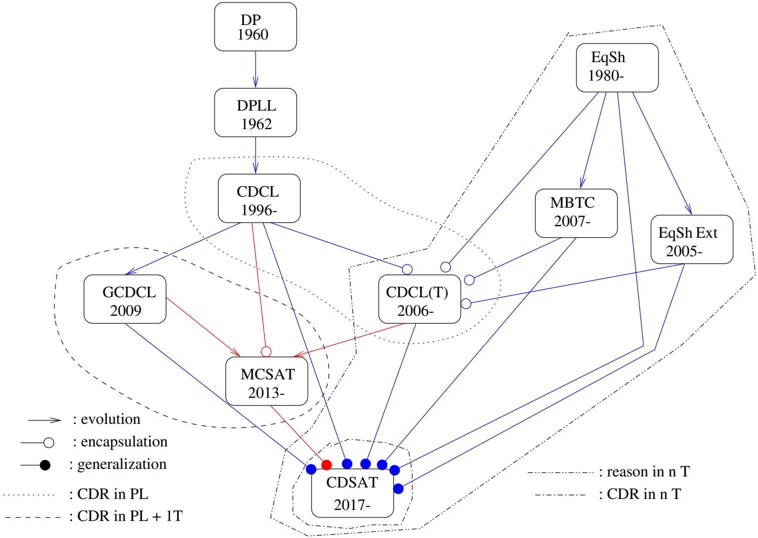
- Assume $y \prec_{\mathsf{LRA}} x$
- 2nd FM-resolution inference in the non-halting sequence:
  $\{x < -y,\ -y < -2\} \vdash_{\mathsf{LRA}} x < -2$
  is barred: it resolves on y when x occurs in the premises
- All GCDCL or CDSAT derivations embedding that diverging series of FM-resolution inferences are barred
- Multiple CDSAT-derivations discover that
  $l_0\colon\ -2{\cdot}x - y < 0,\ l_1\colon x + y < 0,\ l_2\colon x < -1$
  is LRA-unsatisfiable
- A simple one does it by mere LRA-propagations at level 0

Example where CDSAT emulates GCDCL

$l_0\colon\ -2{\cdot}x - y < 0,\ \ l_1\colon x + y < 0,\ \ l_2\colon x < -1$ (level 0)
Assume $y \prec_{\mathrm{LRA}} x$

1. Deduce: $l_3\colon -y < -2$ (level 0)
   $l_0 + 2l_2\colon\quad \{-y < 2{\cdot}x,\ 2{\cdot}x < -2\} \vdash_{\mathrm{LRA}} -y < -2$
   /* x is $\prec_{\mathrm{LRA}}$-max variable in both premises */
2. Deduce: $l_4\colon y < 0$ (level 0) /*normal form of $-y < -2{\cdot}y$ */
   $l_0 + 2l_1\colon\quad \{-y < 2{\cdot}x,\ 2{\cdot}x < -2{\cdot}y\} \vdash_{\mathrm{LRA}} -y < -2{\cdot}y$
   /* x is $\prec_{\mathrm{LRA}}$-max variable in both premises */
3. Deduce: $l_5\colon 2 < 0$ (level 0)
   $-l_3 + l_4\colon\quad \{2 < y,\ y < 0\} \vdash_{\mathrm{LRA}} 2 < 0$
   /* y is $\prec_{\mathrm{LRA}}$-max variable in both premises as there is no x*/
4. $\emptyset \vdash_{\mathrm{LRA}} \overline{2 < 0}$ reveals LRA-conflict at level 0
   so that Fail returns unsat

The big picture: better conflict-driven theory reasoning



From GCDCL to MCSAT

Conflict-driven satisfiability procedures for sets of $\mathcal{T}$-literals:

- LIA: Cutting-to-the-chase procedure
  [Jovanović, de Moura: CADE 2011, JAR 2013]
  [Bromberger et al.: CADE 2015]
- NRA: NLSAT
  [Jovanović, de Moura: IJCAR 2012]
- Use first-order assignments
- Explain conflicts by inferences that generate new atoms and may exclude first-order assignments

Conflict-driven satisfiability procedures for sets of $\mathcal{T}$-clauses?

From GCDCL to MCSAT

- No need to generalize to $\mathcal{T}$-clauses an inference rule for $\mathcal{T}$-literals
- Entrust the reasoning about disjunction to CDCL
- Integrate in CDCL a conflict-driven $\mathcal{T}$-satisfiability procedure for sets of $\mathcal{T}$-literals
- CDCL($\mathcal{T}$)? No, it allows
  neither first-order assignment nor new atoms on the trail
  nor $\mathcal{T}$-inferences generating lemmas possibly excluding
  first-order assignments
- MCSAT (Model-Constructing SATisfiability)
  [de Moura, Jovanović: VMCAI 2013]
  [Jovanović, Barrett, de Moura: FMCAD 2013]

MCSAT (Model-Constructing SATisfiability)

- Integrate CDCL and one model-constructing conflict-driven $\mathcal{T}$-sat procedure for sets of $\mathcal{T}$-literals (called $\mathcal{T}$-plugin) that
  - Has access to the trail
  - Proposes assignments to first-order terms: $\mathcal{T}$-assignments
  - Computes $\mathcal{T}$-propagations
  - Explains $\mathcal{T}$-conflicts by $\mathcal{T}$-inferences generating lemmas possibly excluding $\mathcal{T}$-assignments
  - Lemma may contain new (i.e., non-input) atoms coming from a finite basis for termination
- CDCL and the $\mathcal{T}$-plugin cooperate in model construction
- Both propositional and $\mathcal{T}$-reasoning are conflict-driven

CDSAT generalizes MCSAT

- CDSAT generalizes MCSAT to generic union $\mathcal{T} = \bigcup_{k=1}^{n} \mathcal{T}_k$
- MCSAT is not a combination calculus
  hence does not cover, e.g.:
  - Interaction of multiple first-order theories on the trail
  - Conflict-drivenness for more than one first-order theory
  - Combination of conflict-driven and black-box procedures
  - Soundness, completeness, termination for theory combination
  - Construction of finite global basis from local ones
- CDSAT does not require model-constructing $\mathcal{T}_k$-sat
  procedures in the sense of MCSAT

CDSAT generalizes MCSAT

- CDSAT and MCSAT have different transition systems, e.g.:
  - MCSAT evaluation mechanism $\rightsquigarrow$ $\mathcal{T}_k$-inferences in CDSAT
  - MCSAT explanation function $\rightsquigarrow$ $\mathcal{T}_k$-inferences in CDSAT
    explanation function: private to $\mathcal{T}_k$-plugin
    $\mathcal{T}_k$-inferences in CDSAT: public on shared trail
- CDSAT provides foundations for instances of theory combination in MCSAT implementations, e.g.:
  Bool $\cup$ EUF $\cup$ LRA [Jovanović, Barrett, de Moura: FMCAD 2013]
- CDSAT allows predicate-sharing theories
  MCSAT assumes disjoint theories

CDSAT reduces to MCSAT if theory union contains only Bool and one theory $\mathcal{T}$ equipped with a conflict-driven model-constructing $\mathcal{T}$-sat procedure for sets of $\mathcal{T}$-literals

Example where CDSAT emulates MCSAT

$x < y,\ x < z,\ (y < w) \vee (z < w),\ w < x$ (level 0)
Assume $x \prec_{\text{LRA}} y \prec_{\text{LRA}} z \prec_{\text{LRA}} w$ and a sensible search plan

1. Decide: ${}_?(x \leftarrow 0)$ (level 1) /* acceptable */
2. Decide: ${}_?(y \leftarrow 1)$ (level 2) /* acceptable */
   /* ${}_?(y \leftarrow 0)$ not acceptable: $\{x \leftarrow 0,\ y \leftarrow 0\} \vdash_{\text{LRA}} \overline{(x < y)}$*/
3. Decide: ${}_?(z \leftarrow 1)$ (level 3) /* acceptable */
   /* ${}_?(z \leftarrow 0)$ not acceptable: $\{x \leftarrow 0,\ z \leftarrow 0\} \vdash_{\text{LRA}} \overline{(x < z)}$*/
   LRA-conflict:
   $\{x \leftarrow 0,\ y \leftarrow 1,\ z \leftarrow 1,\ w < x,\ (y < w) \vee (z < w)\}$
   Equivalently: no acceptable value for w
   Disjunction: case analysis by Bool-module

Example where CDSAT emulates MCSAT

4. Decide: ${}_{?}(y < w)$ (level 4)
5. Deduce: ${}_{J\vdash}(y < x)$ (level 4)
   $J = \{{}_{?}(y < w),\ {}_{\emptyset\vdash}(w < x)\}$ (level 4)
   $\{{}_{?}(y < w),\ {}_{\emptyset\vdash}(w < x)\} \vdash_{\mathsf{LRA}} y < x$
   /* w is $\prec_{\mathsf{LRA}}$-max variable in both $y < w$ and $w < x$ */
6. Deduce: ${}_{I\vdash}(x < x)$ (level 4)
   $I = \{{}_{\emptyset\vdash}(x < y),\ {}_{J\vdash}(y < x)\}$ (level 4)
   $\{{}_{\emptyset\vdash}(x < y),\ {}_{J\vdash}(y < x)\} \vdash_{\mathsf{LRA}} x < x$
   /* y is $\prec_{\mathsf{LRA}}$-max variable in both $x < y$ and $y < x$ */
   LRA-conflict: $E_0 = \{{}_{I\vdash}(x < x)\}$
7. Resolve: $E_1 = \{{}_{\emptyset\vdash}(x < y),\ {}_{J\vdash}(y < x)\}$
8. Resolve: $E_2 = \{{}_{\emptyset\vdash}(x < y),\ {}_{?}(y < w),\ {}_{\emptyset\vdash}(w < x)\}$

Example where CDSAT emulates MCSAT

9. LearnBackjump: back to level 0 adding ${}_{H\vdash}(\overline{y < w})$
   $H = \{{}_{\emptyset\vdash}(x < y),\ {}_{\emptyset\vdash}(w < x)\}$
   /* 0 is smallest level where $\overline{y < w}$ is undefined */
10. Deduce: ${}_{G\vdash}(z < w)$ (level 0)
    $G = \{{}_{H\vdash}(\overline{y < w}),\ {}_{\emptyset\vdash}((y < w) \vee (z < w))\}$ (level 0)
    $\{{}_{H\vdash}(\overline{y < w}),\ {}_{\emptyset\vdash}((y < w) \vee (z < w))\} \vdash_{\text{Bool}} z < w$
    /* shadow rule unnecessary: Bool-module handles $\vee$ by decision and unit propagation; LRA-module reasons about LRA-literals */
11. Deduce: ${}_{K\vdash}(z < x)$ (level 0)
    $K = \{{}_{G\vdash}(z < w),\ {}_{\emptyset\vdash}(w < x)\}$ (level 0)
    $\{{}_{G\vdash}(z < w),\ {}_{\emptyset\vdash}(w < x)\} \vdash_{\text{LRA}} z < x$
    /* w is $\prec_{\text{LRA}}$-max variable in both $z < w$ and $w < x$ */

Example where CDSAT emulates MCSAT

12. Deduce: ${}_{M\vdash}(x < x)$ (level 0)
    $M = \{{}_{\emptyset\vdash}(x < z),\ {}_{\kappa\vdash}(z < x)\}$ (level 0)
    $\{{}_{\emptyset\vdash}(x < z),\ {}_{\kappa\vdash}(z < x)\} \vdash_{\mathsf{LRA}} x < x$
    /* z is $\prec_{\mathsf{LRA}}$-max variable in both $x < z$ and $z < x$ */
13. LRA-conflict: $E_3 = \{{}_{M\vdash}(x < x)\}$ (level 0)
    Fail returns unsat

- Deduce covers both conflict explanation and propagation
- CDSAT can apply inferences (e.g., FM-resolution) more liberally than MCSAT

CDSAT: Conflict-driven reasoning from a theory to many

- Conflict-driven behavior and black-box integration are at odds: each conflict-driven $\mathcal{T}_k$-sat procedure needs to access the trail, post assignments, perform inferences, explain $\mathcal{T}_k$-conflicts, export lemmas
- Key abstraction in CDSAT: open the black-boxes
  pull out the $\mathcal{T}_k$-inference systems
  coordinate them in a conflict-driven way
- If $\mathcal{T}_k$ has no conflict-driven $\mathcal{T}_k$-sat procedure:
  black-box inference rule $L_1, \ldots, L_m \vdash_k \bot$
  invokes the $\mathcal{T}_k$-procedure to detect $\mathcal{T}_k$-unsat

Theory view of an assignment

It defines what a theory sees of an assignment:

- $\mathcal{T}_k$-view of assignment H, written H_k:
  - $\mathcal{T}_k$-assignments in H: those that assign $\mathcal{T}_k$-values
  - $u \simeq t$ if H contains $u \leftarrow \mathfrak{c}$ and $t \leftarrow \mathfrak{c}$
  - $u \not\simeq t$ if H contains $u \leftarrow \mathfrak{c}$ and $t \leftarrow \mathfrak{q}$ with $\mathfrak{c} \neq \mathfrak{q}$

  u and t of $\mathcal{T}_k$-sort s ($s \neq \mathrm{prop}$)

  $u \leftarrow \mathfrak{c}$ and $t \leftarrow \mathfrak{c}$ may be posted by $\mathcal{T}_j$ ($k \neq j$) sharing s
- Global view:
  - The $\mathcal{T}$-view of H for $\mathcal{T} = \bigcup_{k=1}^{n} \mathcal{T}_k$
  - $H_{\mathcal{T}}$ has everything

Key notion for theory combination (MCSAT does not have it)

Theory view: example

$$H = \{x > 1,\ \mathrm{store}(a, i, v) \simeq b,\ \mathrm{select}(a, j) \leftarrow \mathrm{red},\ y \leftarrow -1,\ z \leftarrow 2\}$$

- $H_{\mathrm{Bool}} = \{x > 1,\ store(a, i, v) \simeq b\}$
- $H_{\mathrm{Arr}} = \{x > 1,\ store(a, i, v) \simeq b,\ select(a, j) \leftarrow \mathrm{red}\}$
- $H_{\mathrm{LRA}} = \{x > 1,\ store(a, i, v) \simeq b,\ y \leftarrow -1,\ z \leftarrow 2,\ y \not\simeq z\}$
- $H_{\mathrm{EUF}} = \{x > 1,\ store(a, i, v) \simeq b,\ y \not\simeq z\}$
  assuming EUF has the sort Q of the rational numbers
- A Boolean assignment belongs to every theory view
- Global view: $H \cup \{y \not\simeq z\}$

Relevance

Term u is relevant to $\mathcal{T}_k$ in assignment J if

- Either u occurs in J (also as subterm), $\mathcal{T}_k$ has the sort s of u and has values for s
- Term u is an equality $u_1 \simeq_s u_2$ s.t. u_1 and u_2 occur in J, $\mathcal{T}_k$ has sort s, but not values for s
- Term u is a Boolean term $p(u_1, \ldots, u_m)$ s.t. p is a shared predicate symbol (by $\mathcal{T}_k$ and at least another theory), the u_i's occur in J, and $\mathcal{T}_k$ has their sorts

Key notion for theory combination (MCSAT does not have it)

Relevance: example

- $H = \{x \leftarrow 5,\ f(x) \leftarrow 2,\ f(y) \leftarrow 3\}$
- $x, y\colon \mathrm{Q}$, $\quad f\colon \mathrm{Q} \rightarrow \mathrm{Q}$, $\quad$ LRA and EUF share sort Q
- $H_{\mathrm{LRA}} = H \cup \{x \not\simeq f(x),\ x \not\simeq f(y),\ f(x) \not\simeq f(y)\}$
- $H_{\mathrm{EUF}} = \{x \not\simeq f(x),\ x \not\simeq f(y),\ f(x) \not\simeq f(y)\}$
- x and y are LRA-relevant, not EUF-relevant
- $x \simeq y$ is EUF-relevant, not LRA-relevant
- LRA makes x and y equal/different by assigning them same/different values
- EUF makes x and y equal/different by assigning a truth value to $x \simeq y$

Acceptability revisited

$\Gamma_{\mathcal{T}_k}$: the $\mathcal{T}_k$-view of trail Γ

A $\mathcal{T}_k$-assignment $u \leftarrow \mathfrak{c}$ is an acceptable decision ${}_?(u \leftarrow \mathfrak{c})$ for the $\mathcal{T}_k$-module if

1. Term u is relevant to $\mathcal{T}_k$ in $\Gamma_{\mathcal{T}_k}$
2. $\Gamma_{\mathcal{T}_k}$ does not assign a $\mathcal{T}_k$-value to term u
3. If $u \leftarrow \mathfrak{c}$ is a first-order assignment: $t \leftarrow \mathfrak{c}$ does not trigger a $\mathcal{T}_k$-inference $J \cup \{u \leftarrow \mathfrak{c}\} \vdash_k \overline{L}$ with $J \subseteq \Gamma_{\mathcal{T}_k}$ and $L \in \Gamma_{\mathcal{T}_k}$

CDSAT transition rule UndoDecide

- The assignment of max level in conflict E is a justified assignment ${}_{J\vdash}L$ where J contains a first-order decision ${}_{?}A$ such that $\mathrm{level}_{\Gamma}({}_{?}A) = \mathrm{level}_{\Gamma}(J) = \mathrm{level}_{\Gamma}(E)$
- UndoDecide undoes ${}_{?}A$, backtracks, and puts $\overline{L}$ on the trail
- A first-order assignment does not have a complement, but its Boolean consequence does
- Resolve is forbidden: replacing ${}_{J\vdash}L$ with J in E and undoing ${}_{?}A$ by UndoClear can cause a loop if Decide reiterates ${}_{?}A$

CDSAT module for arrays

- Signature Σ_{Arr}:
  - Sorts: $S = \{\mathrm{prop}, \mathrm{I}, \mathrm{V}, \mathrm{A}\}$, I: indices, V: (array) values, A: arrays with indices of sort I and values of sort V
  - Symbols: $\simeq_s$ for all $s \in S$, select (read), store (write)
- Theory extension Arr^+ may be trivial or add countably many values for each $s \in S \setminus \{\mathrm{prop}\}$
- Inference rules corresponding to the select-over-store axioms:
  1. $i \simeq j \longrightarrow \mathrm{select}(\mathrm{store}(a, i, v), j) \simeq v$
     $\{i \simeq j,\ b \simeq \mathrm{store}(a, i, v),\ \mathrm{select}(b, j) \not\simeq v\} \vdash_{\mathrm{Arr}} \bot$
  2. $i \not\simeq j \longrightarrow \mathrm{select}(\mathrm{store}(a, i, v), j) \simeq \mathrm{select}(a, j)$
     $\{i \not\simeq j,\ b \simeq \mathrm{store}(a, i, v),\ \mathrm{select}(b, j) \not\simeq \mathrm{select}(a, j)\} \vdash_{\mathrm{Arr}} \bot$

CDSAT module for arrays with extensionality

- Extensionality axiom:
  $(\forall i.\ \mathsf{select}(a,i) \simeq \mathsf{select}(b,i)) \longrightarrow a \simeq b$
- Clausal form:
  $\mathsf{select}(a, \mathsf{diff}(a,b)) \not\simeq \mathsf{select}(b, \mathsf{diff}(a,b)) \vee a \simeq b$
  Skolem function $\mathsf{diff} : \mathsf{A} \times \mathsf{A} \to \mathsf{I}$ captures the witness index
- Inference rule:
  $a \not\simeq b \vdash_{\mathsf{Arr}} \mathsf{select}(a, \mathsf{diff}(a,b)) \not\simeq \mathsf{select}(b, \mathsf{diff}(a,b))$
- $\mathsf{basis}_{\mathsf{Arr}}(X)$: all subterms of terms in X, equalities btw them, and witness terms $\mathsf{select}(a, \mathsf{diff}(a,b))$, $\mathsf{select}(b, \mathsf{diff}(a,b))$

Example with theory clauses and UndoDecide

- Input set S contains clauses:
  - C_1: $(i \not\simeq j) \vee (\mathsf{select}(\mathsf{store}(a, i, v), j) < \mathsf{select}(a, j))$
  - C_2: $(\mathsf{select}(a, j) - \mathsf{select}(a, k)) \simeq 0$
  - C_3: $(\mathsf{select}(\mathsf{store}(a, i, v), j) \not< \mathsf{select}(a, j)) \vee$ $(\mathsf{select}(a, j) + \mathsf{select}(a, k) \simeq v)$
- Theory union: $\mathsf{Bool} \cup \mathsf{LRA} \cup \mathsf{Arr}$
- Suppose Arr interprets indices as integers:
  $\mathsf{I} = \mathsf{Z}$ and Arr^+ adds integer constants as Arr-values

Example with theory clauses and UndoDecide

1. Arr-module: Decide $_?(i{\leftarrow}0)$ (level 1)
   /* acceptable as i is relevant to Arr */
2. Arr-module: Decide $_?(j{\leftarrow}0)$ (level 2)
3. Arr-module: equality inference $\{i{\leftarrow}0, j{\leftarrow}0\} \vdash_{\mathrm{Arr}} i \simeq j$
   Deduce: $A_1 : {}_{J}{\vdash}(i \simeq j)$ with $J = \{_?(i{\leftarrow}0), {}_?(j{\leftarrow}0)\}$ (level 2)
4. Bool-module: unit propagation
   $\{A_1,\ C_1\} \vdash_{\mathrm{Bool}} \mathrm{select}(\mathrm{store}(a, i, v), j) < \mathrm{select}(a, j)$
   Deduce: $A_2 : {}_{I}{\vdash}(\mathrm{select}(\mathrm{store}(a, i, v), j) < \mathrm{select}(a, j))$
   with $I = \{A_1,\ C_1\}$ (level 2)

Example with theory clauses and UndoDecide

5. Bool-module: unit propagation
   $\{A_2,\ C_3\} \vdash_{\mathsf{Bool}} \mathrm{select}(a,j) + \mathrm{select}(a,k) \simeq v$
   Deduce: $A_3\colon {}_{H\vdash}(\mathrm{select}(a,j) + \mathrm{select}(a,k) \simeq v)$
   with $H = \{A_2,\ C_3\}$ (level 2)
6. Arr-module: first select-over-store rule
   $\{A_1,\ A_2\} \vdash_{\mathsf{Arr}} v < \mathrm{select}(a,j)$
   Deduce: $A_4\colon {}_{G\vdash}(v < \mathrm{select}(a,j))$
   with $G = \{A_1,\ A_2\}$ (level 2)
7. LRA-module: FM-resolution $A_3 + C_2$
   $\{A_3,\ C_2\} \vdash_{\mathsf{LRA}} \mathrm{select}(a,j) \simeq v/2$
   Deduce: $A_5\colon {}_{M\vdash}(\mathrm{select}(a,j) \simeq v/2)$
   with $M = \{A_3,\ C_2\}$ (level 2)

Example with theory clauses and UndoDecide

LRA-conflict: $E_0 = \{A_4,\ A_5\}$
as $A_4 :\ _{G}\vdash(v < \text{select}(a, j))$ and $A_5 :\ _{M}\vdash(\text{select}(a, j) \simeq v/2)$

8. E_0 contains literals A_4 and A_5 of max level (2)
   Resolve: $E_1 = \{A_4,\ A_3,\ C_2\}$
9. E_1 contains literals A_3 and A_4 of max level (2)
   Resolve: $E_2 = \{A_1,\ A_2,\ A_3,\ C_2\}$
10. E_2 contains literals A_1, A_2 and A_3 of max level (2)
    Resolve: $E_3 = \{A_1,\ A_2,\ C_3,\ C_2\}$
11. E_3 contains literals A_1, and A_2 of max level (2)
    Resolve: $E_4 = \{A_1,\ C_1,\ C_3,\ C_2\}$

Example with theory clauses and UndoDecide

$E_4 = \{A_1,\ C_1,\ C_3,\ C_2\}$

E_4 contains one literal of max level: $\text{level}_\Gamma(A_1) = 2 = \text{level}_\Gamma(E_4)$

A_1 is ${}_{J\vdash}(i \simeq j)$ and $J = \{{}_?(i{\leftarrow}0), {}_?(j{\leftarrow}0)\}$
where ${}_?(j{\leftarrow}0)$ also has level 2

Apply Resolve to replace A_1 with J
and UndoClear to undo ${}_?(j{\leftarrow}0)$?

No, the system could loop by repeating ${}_?(j{\leftarrow}0)$
(still acceptable)

Example with theory clauses and UndoDecide

12. UndoDecide: undo $_?(j \leftarrow 0)$, backtrack to level 1, and add decision $_?(i \not\simeq j)$ (level 2)
    /* C_1: $(i \not\simeq j) \vee (\text{select}(\text{store}(a, i, v), j) < \text{select}(a, j))$ is satisfied */
13. LRA-module: Decide $_?(\text{select}(a, j) \leftarrow 1)$ (level 3)
14. LRA-module: Decide $_?(\text{select}(a, k) \leftarrow 1)$ (level 4)
    /* C_2: $(\text{select}(a, j) - \text{select}(a, k)) \simeq 0$ is satisfied */
15. LRA-module: Decide $_?(v \leftarrow 2)$ (level 5)
    /* C_3: $(\text{select}(\text{store}(a, i, v), j) \not< \text{select}(a, j)) \vee$ $(\text{select}(a, j) + \text{select}(a, k) \simeq v)$ is satisfied */

Example with theory clauses and UndoDecide: variant

Suppose theory Arr does not have values for array indices:
i and j not relevant, Arr-module cannot decide their values

1. Arr-module: Decide $_?(i \simeq j)$ (level 1)
   /* acceptable as $i \simeq j$ is relevant to Arr */
2. The same transitions as before lead to conflict
   $\{_?(i \simeq j),\ C_1,\ C_3,\ C_2\}$ (level 1)
3. LearnBackjump backtracks to level 0 and places $_{N\vdash}(i \not\simeq j)$ on the trail with $N = \{C_1,\ C_3,\ C_2\}$
4. The satisfiability of the clauses can be detected as before

Current and future work

- More theory modules: maps, vectors (aka dynamic arrays), vectors with concatenation
  (possibly subsuming sequences and hence strings)
- Formulas with quantifiers: CDSAT(QSMA)
- CDSAT search plans: both global and local issues
  - Heuristic strategies to make decisions, prioritize theory inferences, control lemma learning
  - Efficient techniques to detect applicability of theory inference rules and acceptability of decisions
- Architecture of a CDSAT solver
- Baby verified implementation written in Rust by Xavier Denis:
  `https://github.com/xldenis/cdsat`

Selected references

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- CDSAT for predicate-sharing theories: arrays, maps, and vectors with abstract domain. In preparation.

Thank you!