Rewrite-based decision procedures

Maria Paola Bonacina

Dipartimento di Informatica
Università degli Studi di Verona
Verona, Italy, EU

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Modularity of termination for combination of theories

Modularity of termination:
if $\mathcal{SP}_{\succ}$-strategy terminates on $\mathcal{T}_i$-sat problems then it terminates on $\mathcal{T}$-sat problems for $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i$.

Hypotheses:

- No shared function symbols (shared constants allowed)
- *Variable-inactive* theories

Variable-inactivity

Clause C *variable-inactive*: no maximal literal in C is equation $t \simeq x$ where $x \notin Var(t)$

Set of clauses *variable-inactive*: all its clauses are

$\mathcal{T}$ *variable-inactive*: the limit $S_\infty = \bigcup_{j\geq 0} \bigcap_{i\geq j} S_i$ of a fair derivation from $\mathcal{T} \cup S$ is variable-inactive

Examples

$$
\begin{aligned}
C_1 &= \mathrm{car}(\mathrm{cons}(x, y)) \simeq x \\
C_2 &= z \simeq w \vee \mathrm{select}(\mathrm{store}(x, z, v), w) \simeq \mathrm{select}(x, w) \\
C_3 &= \bigvee_{1 \leq j < k \leq n} (x_j \simeq x_k)
\end{aligned}
$$

C_1 variable-inactive
C_2 variable-inactive
C_3 not variable-inactive (*cardinality constraint clause*)

The modularity theorem

Theorem: if

- $\mathcal{T}_i$, $1 \leq i \leq n$, do not share function symbols
- $\mathcal{T}_i$, $1 \leq i \leq n$, variable-inactive
- $\mathcal{SP}_{\succ}$-strategy is a $\mathcal{T}_i$-satisfiability procedure, $1 \leq i \leq n$,

then it is a $\mathcal{T}$-satisfiability procedure for $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i$.

All theories considered so far are variable-inactive.

Explanation of the proof of the theorem

- No shared function symbol: no paramodulation from compound terms across theories
- Variable-inactivity: no paramodulation from variables across theories, since for $t \simeq x$ where $x \in Var(t)$ it is $t \succ x$

Only paramodulations from constants into constants: finitely many.

Comment on shared function symbols

- If $\mathcal{T}_1$ contains an axiom where f occurs and $\mathcal{T}_2$ contains another axiom where f occurs, we may have all possible inferences between two general clauses, of whom we know no special properties or restrictions.
- The symbols from the theories appear freely mixed in S, and are separated by flattening (does the job of "purification").

Variable-inactive theories

- *Purely equational theories*:
  no trivial models $\Rightarrow$ variable-inactive
- *Horn theories*:
  no trivial models + maximal unit strategy $\Rightarrow$ variable-inactive
- *Maximal unit strategy*:
  restricts superposition to unit clauses and paramodulates unit clauses into maximal negative literals

Variable inactivity and stable-infiniteness

Lemma: If S_0 is a finite satisfiable set of clauses, then S_0 admits no infinite models if and only if the limit S_∞ of any fair $\mathcal{SP}_\succ$-derivation from S_0 contains a cardinality constraint clause.

Theorem: If $\mathcal{T}$ is variable-inactive, then it is stably-infinite.

Lemma from:
Maria Paola Bonacina, Silvio Ghilardi, Enrica Nicolini, Silvio Ranise and Daniele Zucchelli.
Decidability and undecidability results for Nelson-Oppen and rewrite-based decision procedures.

Proc. 3rd IJCAR, LNAI 4130:513-527, Springer 2006.

$\mathcal{T}$-decision procedure

$\mathcal{T}$-**decision procedure**: decide satisfiability of a *conjunction of ground clauses* in theory $\mathcal{T}$

S: *set of ground clauses* in the signature of $\mathcal{T}$

$\mathcal{T}$: *presentation* of a theory

$\bowtie$ is either $\simeq$ or $\not\simeq$

Flat and strictly flat

Depth:
$depth(t) = 0$, if t is constant or variable
$depth(t) = 1 + max\{depth(t_i) : 1 \leq i \leq n\}$, if t is $f(t_1, \ldots, t_n)$
$depth(l \bowtie r) = depth(l) + depth(r)$

Term: t is *flat* if $depth(t) \leq 1$, *strictly flat* if $depth(t) = 0$

Literal: $l \simeq r$ is *flat* if $depth(l \simeq r) \leq 1$
$l \not\simeq r$ is *flat* if $depth(l \not\simeq r) = 0$
$l \bowtie r$ is *strictly flat* if $depth(l \bowtie r) = 0$

Clause: C is *(strictly) flat* if all its literals are
$Maxd(C) = max\{depth(t) : t\ occurs\ in\ C\}$

Flattening

S: given set of ground clauses

S': flattened version of S such that

- all unit clauses in S' are flat
- all non-unit clauses in S' are strictly flat
- $\mathcal{T} \cup S \equiv_{\mathrm{s}} \mathcal{T} \cup S'$, where $\equiv_{\mathrm{s}}$ means equisatisfiable

Example

$$S = \{f(f(a)) \simeq b \vee f(c) \not\simeq d\}$$

$$S' = \{f(a) \simeq c_1,\ f(c_1) \simeq c_2,\ f(c) \simeq c_3,\ c_2 \simeq b \vee c_3 \not\simeq d\}$$

"Good" CSO

- Simplification ordering
- *Complete*: *total* on ground terms
- *"Good"*: $t \succ c$ for all ground compound terms t and constants c

Thus, we drop requirements such as $a \succ e \succ i$ for arrays.

Intuition

In a $\mathcal{T}$-decision problem we distinguish:

- $\mathcal{T}_g$: ground clauses
- $\mathcal{T}_1$: non-ground clauses about properties that can be deduced using one interpreted function
- $\mathcal{T}_2$: non-ground clauses about the interaction of two interpreted functions

Example: Arrays

$$\begin{array}{rc} \forall x, z, v. & \mathrm{select}(\mathrm{store}(x, z, v), z) \simeq v \\ \forall x, z, w, v. & z \not\simeq w \supset \mathrm{select}(\mathrm{store}(x, z, v), w) \simeq \mathrm{select}(x, w) \\ \forall x, y. & \forall z.\, \mathrm{select}(x, z) \simeq \mathrm{select}(y, z) \supset x \simeq y \end{array}$$

First two axioms: in $\mathcal{T}_2$
Third axiom (*extensionality*): in $\mathcal{T}_1$

Subterm-inactivity of a 3-tuple

$\mathcal{T}_g, \mathcal{T}_1, \mathcal{T}_2$: disjoint sets of clauses

$\langle \mathcal{T}_g, \mathcal{T}_1, \mathcal{T}_2 \rangle$ is *subterm-inactive* if

- $\mathcal{T}_g$ is ground and flattened
- $\mathcal{T}_1$ is interaction-free from $\mathcal{T}_2$ and satisfies certain closure properties
- $\mathcal{T}_2$ is saturated and satisfies certain closure properties

Subterm-inactivity of a set

Set P is *subterm-inactive* if $P = \mathcal{T}_g \uplus \mathcal{T}_1 \uplus \mathcal{T}_2$ such that $\langle \mathcal{T}_g, \mathcal{T}_1, \mathcal{T}_2 \rangle$ is subterm-inactive.

Say P is presentation $\mathcal{T}$: typically $\mathcal{T}_g = \emptyset$.

If presentation $\mathcal{T}$ is subterm-inactive, then $\mathcal{T} \cup S$, where S is ground and flattened, is also: $\mathcal{T} \cup S = S \uplus \mathcal{T}_1 \uplus \mathcal{T}_2$

The subterm-inactivity theorem

If $\mathcal{T}_g \uplus \mathcal{T}_1 \uplus \mathcal{T}_2$ is *subterm-inactive*, then:

- For all persistent clauses D generated by $\mathcal{SP}$:
  - $Maxd(D) \leq max\{Maxd(C) \colon C \; premise\}$ (depth-preserving)
  - If D is ground, $\langle \mathcal{T}_g \cup \{D\}, \mathcal{T}_1, \mathcal{T}_2 \rangle$ is *subterm-inactive*
  - If D is not ground, $\langle \mathcal{T}_g, \mathcal{T}_1 \cup \{D\}, \mathcal{T}_2 \rangle$ is *subterm-inactive*
- $\mathcal{SP}_{\succ}$-strategy is $\mathcal{T}$-decision procedure
- $\mathcal{T}$ is *variable-inactive* (hence easy to combine)

Impact of subterm-inactivity on binary inferences

- An $\mathcal{SP}$-inference between two clauses in $\mathcal{T}_1$ generates a clause in $\mathcal{T}_1$ or $\mathcal{T}_g$ (Closure properties)
- An $\mathcal{SP}$-inference between two clauses in $\mathcal{T}_2$ generates a clause that is deleted eventually (Saturation)
- No $\mathcal{SP}$-inference applies to a clause in $\mathcal{T}_1$ and a clause in $\mathcal{T}_2$ (Interaction-freeness)
- An $\mathcal{SP}$-inference between a clause in $\mathcal{T}_1 \cup \mathcal{T}_2$ and a clause in $\mathcal{T}_g$ generates a clause in $\mathcal{T}_1$ or $\mathcal{T}_g$ (Closure properties + flatness)

Impact of subterm-inactivity on unary inferences

- An $\mathcal{SP}$-inference from a clause in $\mathcal{T}_1$ generates a clause that is in $\mathcal{T}_1$ or in $\mathcal{T}_g$ or is deleted eventually (Closure properties)
- An $\mathcal{SP}$-inference from a clause in $\mathcal{T}_2$ generates a clause that is deleted eventually (Saturation)

Subterm-inactive theories

- Equality
- Arrays with or without extensionality + two variations
- Recursive data structures (including integer offsets and acyclic lists)
- Finite sets with or without extensionality

Not included: Records, integer offsets modulo, possibly empty possibly cyclic lists

Variations on the theory of arrays

- Add to $\mathcal{A}$ an *injectivity predicate* to state than an array is injective:
  $Inj(x) \Leftrightarrow \forall z, w.\ (z \not\simeq w \supset \mathrm{select}(x, z) \not\simeq \mathrm{select}(x, w))$
- Add to $\mathcal{A}$ a *swap predicate*:
  $Swap(x, y, z_1, z_2) \Leftrightarrow$
  $\mathrm{select}(x, z_1) \simeq \mathrm{select}(y, z_2) \wedge$
  $\mathrm{select}(x, z_2) \simeq \mathrm{select}(y, z_1) \wedge$
  $\forall w.\ (w \not\simeq z_1 \wedge w \not\simeq z_2 \supset \mathrm{select}(x, w) \simeq \mathrm{select}(y, w))$

Finite sets

$$
\begin{array}{rc}
\forall x, v. & \mathrm{member}(v, \mathrm{insert}(v, x)) \simeq \mathit{true} \\
\forall x, v, w. & v \not\simeq w \supset \mathrm{member}(v, \mathrm{insert}(w, x)) \simeq \mathrm{member}(v, x) \\
\forall x, y. & \forall v.\, \mathrm{member}(v, x) \simeq \mathrm{member}(v, y) \supset x \simeq y
\end{array}
$$

First two axioms: $\mathcal{FS}$
With third axiom (*extensionality*): $\mathcal{FS}^e$

A non-obvious example

If we add to $\mathcal{A}$ the axiom:
$Const_y(x) \Leftrightarrow \forall z.\ \mathrm{select}(x, z) \simeq y$
the resulting theory is not subterm-inactive.

It is not variable-inactive either:
$S = \{store(a, i, e_1) \simeq a',\ Const_e(a),\ Const_{e'}(a')\} \Leftrightarrow$
$\{store(a, i, e_1) \simeq a',\ \mathrm{select}(a, z) \simeq e,\ \mathrm{select}(a', z) \simeq e'\}$

Superposition of $store(a, i, e_1) \simeq a'$ into axiom
$z \simeq w \vee \mathrm{select}(\mathrm{store}(x, z, v), w) \simeq \mathrm{select}(x, w)$
generates $w \simeq i \vee \mathrm{select}(a, w) \simeq \mathrm{select}(a', w)$
which is simplified to
$w \simeq i \vee e \simeq e'$
which is not variable-inactive because $w \simeq i$ is maximal.

Discussion

- Termination results for T-satisfiability procedures were obtained by analyzing all possible $\mathcal{SP}$-inferences
- Subterm-inactivity is obtained by generalizing those analyses
- Its conditions are syntactic and most of them could be tested automatically
- However, they are very complicated, inter-twined and not intuitive
- Simpler, hence better, approach: T-decision procedures assuming only *variable-inactivity*

Flattening again

S: given set of ground clauses

$S_1 \uplus S_2$: flattened version of S such that

- S_1: unit flat clauses
- S_2: strictly flat non-unit clauses
- $\mathcal{T} \cup S \equiv_s \mathcal{T} \cup S_1 \cup S_2$, where $\equiv_s$ means equisatisfiable

Example

$$S = \{f(a) \not\simeq f(b) \vee f(a) \not\simeq f(c)\}$$

$$S_1 = \{f(a) \simeq a',\ f(b) \simeq b',\ f(c) \simeq c'\}$$

$$S_2 = \{a' \not\simeq b' \vee a' \not\simeq c'\}$$

where a', b', c' are new constants

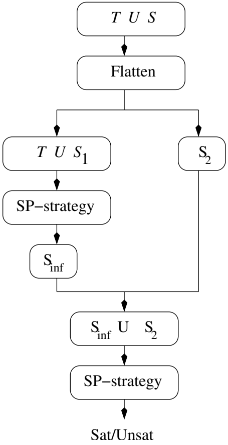
$\mathcal{T}$-decision procedures based on variable-inactivity

Theorem: if

- $\mathcal{T}$ is variable inactive
- $\mathcal{SP}_{\succ}$-strategy is $\mathcal{T}$-satisfiability procedure

then it is also $\mathcal{T}$-decision procedure.

$\mathcal{T}$-decision scheme



Explanation: analysis of inferences

Lemma:
C: variable-inactive clause
C': strictly flat ground clause

1. C' paramodulates into C:
   $C = l[a] \bowtie r \vee D$
   $C' = a \simeq a'' \vee D'$
   Generated clause: $l[a''] \bowtie r \vee D \vee D'$
2. C paramodulates into C':
   $C = a \simeq a' \vee D$ (also strictly flat)
   $C' = a \bowtie a'' \vee D'$
   Generated clause: $a' \bowtie a'' \vee D \vee D'$ (also strictly flat)

In both cases mgu is empty.

Proof of the $\mathcal{T}$-decision theorem

- S_∞ limit of derivation from $\mathcal{T} \cup S_1$ is
  - finite
  - variable-inactive
- S_2 is strictly flat
- All inferences between S_∞ and S_2 are paramodulations from constants into constants: finitely many in a finite signature

Putting it all together

- Variable-inactivity is the fundamental requirement for both
  - Combination of theories
  - Generalization to $\mathcal{T}$-decision problems
- The $\mathcal{T}$-decision scheme applies also when $\mathcal{T}$ is a union of variable-inactive theories
- The two applications of $\mathcal{SP}$ are only for clarity: if S_∞ limit of derivation from $\mathcal{T} \cup S_1$ is finite and variable-inactive, it will be such also in a single run from $\mathcal{T} \cup S_1 \cup S_2$, and it will have only finitely many inferences with strictly flat S_2.

Decomposition: unite FOL+= and SMT strengths

- *Decomposition*: definitional and operational part
- *Theory compilation*: apply FOL+= prover to "compile" the definitional part: theory reasoning, non-ground equational reasoning
- *Decision*: apply SMT-solver to subset of saturated set (without $\mathcal{T}$-axioms) + operational part
- Sufficient conditions to preserve satisfiability

Decomposition

Decomposition: generalization of flattening, where S is decomposed into S_1 and S_2; it suffices that S_1 be made of *flat unit clauses*.

- **Records**: S_1 contains the clauses $\mathrm{rstore}_i(a, e) \simeq b$ and S_2 contains everything else
- **Integer offsets**: same as flattening
- **Arrays**: S_1 contains the clauses $\mathrm{store}(a, i, e) \simeq a'$ and S_2 contains everything else

Framework of sufficient conditions

$\mathcal{T}$-compatibility: A is $\mathcal{T}$-compatible with S if A entails every clause generated from premise in S and premise in $\mathcal{T}$

Theorem: $\bar{S}$ is $\mathcal{T}$-compatible with S where $S_\infty = \mathcal{T} \cup \bar{S}$ is the limit generated by $\mathcal{SP}$ from $\mathcal{T} \cup S$

$\mathcal{T}$-stability: ensures that $\mathcal{T}$-compatibility is preserved by all inferences:
if A is $\mathcal{T}$-compatible with S and $\mathcal{T} \cup S \vdash \mathcal{T} \cup S'$ then A is $\mathcal{T}$-compatible with S'.

$\mathcal{T}$-decision by stages: the main theorem

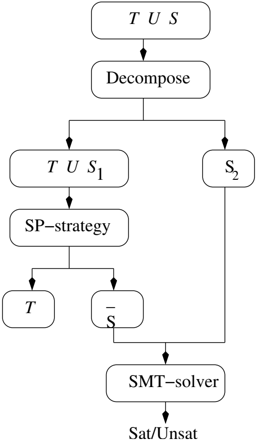
Theorem: under $\mathcal{T}$-stability, if A and A' are sets of clauses such that

- $\mathcal{T} \cup S_1 \models A$
- $\mathcal{T} \cup S_2 \models A'$
- A is $\mathcal{T}$-compatible with S_1
- A' is $\mathcal{T}$-compatible with S_2

then $\mathcal{T} \cup S_1 \cup S_2 \equiv_{\mathrm{s}} A \cup A'$.

Instance of the theorem:A' is S_2 itself; A is $\bar{S}$ where $S_\infty = \mathcal{T} \cup \bar{S}$ is the limit generated by $\mathcal{SP}$ from $\mathcal{T} \cup S_1$

$\mathcal{T}$-decision by stages



Application to the theories

- **Records**: $\bar{S} \cup S_2$ is ground: its satisfiability can be decided by decision procedure for equality (reduction to EUF)
- **Integer offsets**: same as for records
- **Arrays**: $\bar{S} \cup S_2$ is not ground: $\bar{S}$ contains $\mathrm{select}(a, x) \simeq \mathrm{select}(a', x) \vee x \simeq i_1 \vee \ldots \vee x \simeq i_n \vee B$
  where B is possibly empty, ground, strictly flat
  It falls in the *array property fragment*.

Postponing theories

How about theories such as arithmetic or bitvectors that do not lend themselves to general first-order deduction?

Those parts of the problem can be left into S_2 and passed on directly to the SMT-solver.

References

- Alessandro Armando, Maria Paola Bonacina, Silvio Ranise and Stephan Schulz. *New results on rewrite-based satisfiability procedures*. *ACM Trans. on Computational Logic*, To appear. (Presented in part at *FroCoS 2005* and *PDPAR 2005*)
- Maria Paola Bonacina and Mnacho Echenim. *Rewrite-based decision procedures*. *Proc. 6th STRATEGIES Workshop, FLoC 2006*, ENTCS 174(11):27-45, Elsevier 2007.
- Maria Paola Bonacina and Mnacho Echenim. *On variable-inactivity and polynomial T-satisfiability procedures*. *Journal of Logic and Computation*, 18(1): 77-96, Feb. 2008. (Presented in part at *PDPAR 2006*)
- Maria Paola Bonacina and Mnacho Echenim. *Theory decision by decomposition*. Submitted to journal, April 2008. (Presented in part at *CADE 2007*)