Canonical Inference for Implicational Systems

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Motivation

- Knowledge compilation: make efficient reasoning possible
- Completion of equational theories:
  - Canonical presentation
  - Normal-form proofs
- Implicational systems: simple and relevant (e.g., relational databases, abstract interpretations)
- Computing with an implicational system: applying a closure operator or computing minimal model
- **Question:** investigate canonicity of implicational systems

Implicational systems

V: vocabulary of propositional variables

Implicational system S: a set of implications

$$S = \{a_1 \cdots a_n \Rightarrow c_1 \cdots c_m : a_i, c_j \in V\}$$

where antecedent and consequent are conjunctions of (distinct) propositions

Notation: $A \Rightarrow_S B$ for $A \Rightarrow B \in S$

Example

$$S = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow a\}$$

Unary implicational system: all its implications are *unary*, e.g., $ac \Rightarrow d$

A *non-negative Horn clause* is a *unary implication* and vice-versa

Non-unary implications can be decomposed, e.g.:
$a \Rightarrow bf$ into $a \Rightarrow b$ and $a \Rightarrow f$

Consider only unary implicational systems

Moore families

V: vocabulary of propositional variables

Moore family $\mathcal{F}$: a family of subsets of V

- that contains V and
- is closed under intersection

A subset $X \subseteq V$ represents a propositional interpretation
A Moore family is a family of models:

$$\text{Moore families} \sim \text{Horn theories}$$

Closure operators

Moore families $\sim$ Closure operators

Closure operator $\varphi\colon \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ is an operator that is

- *monotone*: $X \subseteq X'$ implies $\varphi(X) \subseteq \varphi(X')$
- *extensive*: $X \subseteq \varphi(X)$
- *idempotent*: $\varphi(\varphi(X)) = \varphi(X)$

Moore families and closure operators

Given φ, its *associated Moore family* $\mathcal{F}_\varphi$ is the set of its fixed points:

$$\mathcal{F}_\varphi = \{X \subseteq V : X = \varphi(X)\}$$

Given $\mathcal{F}$, its *associated closure operator* $\varphi_\mathcal{F}$ maps $X \subseteq V$ to the least element of $\mathcal{F}$ that contains X:

$$\varphi_\mathcal{F}(X) = \bigcap\{Y \in \mathcal{F} : X \subseteq Y\}$$

Implicational systems, Moore families and closure operators

Given implicational system S

- its *associated Moore family* $\mathcal{F}_S$ is the family of its *models*:

$$\mathcal{F}_S = \{X \subseteq V : X \models S\}$$

- its *associated closure operator* φ_S maps $X \subseteq V$ to the least model of S that satisfies X:

$$\varphi_S(X) = \bigcap\{Y \subseteq V : Y \supseteq X \wedge Y \models S\}$$

Computing with an implicational system S:

given X *compute* $\varphi_S(X)$

Example

Implicational system: $S = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow a\}$

Its Moore family:

$$\mathcal{F}_S = \{\emptyset, b, c, d, ab, bc, bd, cd, abd, abe, bcd, abcd, abde, abcde\}$$

Applying its closure operator, e.g.:

$$\varphi_S(ae) = abe$$

Questions

A Moore family : different implicational systems
(In general: a theory may have different presentations)

S and S' such that $\mathcal{F}_S = \mathcal{F}_{S'}$ are *equivalent*

Questions:

- What does it mean for an implicational system to be *canonical*?
- Can we compute canonical implicational systems by appropriate *deduction mechanisms*?

Forward chaining

Given S, $X \subseteq V$, let

$$S(X) = X \cup \bigcup\{B : A \Rightarrow_S B \wedge A \subseteq X\}$$

Then

$$\varphi_S(X) = S^*(X)$$

where

$$S^0(X) = X, \qquad S^{i+1}(X) = S(S^i(X)), \qquad S^*(X) = \bigcup_i S^i(X)$$

Since S, X and V are finite:
$S^*(X) = S^k(X)$ for the smallest k such that $S^{k+1}(X) = S^k(X)$

Example

$$S = \{ac \Rightarrow d, e \Rightarrow a\}$$

$$X = ce$$

$$S(X) = \{ace\}$$

$$S^2(X) = \{acde\}$$

$$\varphi_S(X) = S^*(X) = S^2(X) = \{acde\}$$

Direct implicational system

Intuition:
Direct implicational system:
compute $\varphi_S(X)$ in one single round of *forward chaining*

Definition: S is *direct* if $\varphi_S(X) = S(X)$

Example: $S = \{ac \Rightarrow d, e \Rightarrow a\}$ is *not* direct

[Karell Bertet and Mirabelle Nebut 2004]

Observation

If we have $A \Rightarrow_S B$ and $C \Rightarrow_S D$ such that $A \subseteq X$, $C \not\subseteq X$ and $C \subseteq X \cup B$, more than one iteration of forward chaining is required.

In the example: $e \Rightarrow a$ and $ac \Rightarrow d$ for $X = ce$

To collapse two iterations into one: add $A \cup (C \setminus B) \Rightarrow_S D$

In the example: add $ce \Rightarrow d$

[Karell Bertet and Mirabelle Nebut 2004]

Deduction mechanism: implicational overlap

Implicational overlap

$$\frac{A \Rightarrow BO \quad CO \Rightarrow D}{AC \Rightarrow D}$$

O is the overlap between antecedent and consequent
Conditions:

- $O \neq \emptyset$: there is some overlap
- $B \cap C = \emptyset$: O is all the overlap

Generated direct system

Definition: Given S, the *direct* implicational system $I(S)$ generated from S is the closure of S with respect to implicational overlap.

Theorem: $\varphi_S(X) = I(S)(X)$.

[Karell Bertet and Mirabelle Nebut 2004]

Completion by $\rightsquigarrow_I$ generates direct system

$\rightsquigarrow_I$: deduction mechanism that generates and adds implications by implicational overlap
Note: $\rightsquigarrow_I$ steps are *expansion* steps

Proposition: Given implicational system S for all fair derivations

$$S = S_0 \rightsquigarrow_I S_1 \rightsquigarrow_I \cdots$$

we have

$$S_\infty = I(S)$$

A rewriting-based framework

- Implication $a_1 \cdots a_n \Rightarrow c_1 \cdots c_m$
- Bi-implication $a_1 \cdots a_n c_1 \cdots c_m \Leftrightarrow a_1 \cdots a_n$
- Rewrite rule $a_1 \cdots a_n c_1 \cdots c_m \rightarrow a_1 \cdots a_n$

are equivalent.

Positive literal c: $c \rightarrow true$ ($true$: special constant)

Well-founded ordering $\succ$ on $V \cup \{true\}$ ($true$ minimal) extended by multiset extension.

Associated rewrite system

Given $X \subseteq V$, its *associated rewrite system* is
$R_X = \{x \rightarrow true : x \in X\}$.

Given implicational system S, its *associated rewrite system* is
$R_S = \{AB \rightarrow A : A \Rightarrow_S B\}$.

Given S and X: $R_X^S = R_X \cup R_S$.

Example

$$S = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow a\}$$

$$R_S = \{ab \rightarrow a, acd \rightarrow ac, ae \rightarrow e\}$$

$$X = ae$$

$$R_X = \{a \rightarrow true, e \rightarrow true\}$$

$$R_X^S = \{a \rightarrow true, e \rightarrow true, ab \rightarrow a, acd \rightarrow ac, ae \rightarrow e\}$$

Deduction mechanism: equational overlap

Equational overlap

$$\frac{AO \rightarrow B \quad CO \rightarrow D}{M \rightarrow N} \quad A \cap C = \emptyset \neq O,\ M \succ N$$

O is the overlap between the two left-hand sides:

$$BC \leftarrow AOC \rightarrow AD$$

M and N: *normal-forms* of BC and AD

$\leadsto_E$: deduction mechanism of equational overlap
Note: $\leadsto_E$ features *expansion* and *forward contraction*

Implicational and equational overlap correspond

Intuition: since

- Implicational overlap "unfolds" forward chaining
- Forward chaining is complete for Horn logic

for each non-trivial $\rightsquigarrow_I$ step there is equivalent $\rightsquigarrow_E$ step and vice versa

Lemma: For all implicational systems S,

$$S \rightsquigarrow_I S' \text{ if and only if } R_S \rightsquigarrow_E R_{S'}$$

Example

$S = \{ac \Rightarrow d, e \Rightarrow a\}$

$R_S = \{acd \rightarrow ac, ae \rightarrow e\}$

Implicational overlap yields: $ce \Rightarrow d$

Equational overlap yields:

$$ace \leftarrow acde \rightarrow cde$$

hence

$$cde \rightarrow ce$$

Completion by $\rightsquigarrow_E$ generates direct system

Theorem: For every implicational system S, and for all fair derivations

$$S = S_0 \rightsquigarrow_I S_1 \rightsquigarrow_I \cdots$$

and

$$R_S = R_0 \rightsquigarrow_E R_1 \rightsquigarrow_E \cdots$$

we have

$$R_{(S_\infty)} = (R_S)_\infty$$

Hence $R_{(I(S))} = (R_S)_\infty$

Computing minimal models

Two-stage process:

1. Saturate S w.r.t. implicational overlap to generate $I(S)$
2. For any $X \subseteq V$ compute $\varphi_{I(S)}(X) = \varphi_S(X)$ by forward chaining

One-stage process:

1. Apply completion to R^S_X: output rules $x \rightarrow true$ represent $\varphi_S(X) = \varphi_{I(S)}(X)$

Adding contraction rules

Simplification and Deletion

$$\frac{AC \rightarrow B \quad C \rightarrow D}{AD \rightarrow B \quad C \rightarrow D}\; AD \succ B \qquad \frac{AC \rightarrow B \quad C \rightarrow D}{B \rightarrow AD \quad C \rightarrow D}\; B \succ AD$$

$$\frac{B \rightarrow AC \quad C \rightarrow D}{B \rightarrow AD \quad C \rightarrow D} \qquad \frac{A \leftrightarrow A}{}$$

$\rightsquigarrow_R = \rightsquigarrow_E +$ these rules

Extracting the least model

Theorem: For all $X \subseteq V$, implicational systems S, and fair derivations

$$R_X^S = R_0 \leadsto_R R_1 \leadsto_R \cdots$$

if $Y = \varphi_S(X) = \varphi_{I(S)}(X)$, then

$$R_Y \subseteq (R_X^S)_\infty$$

and

$$R_Y = \{x \to true : x \to true \in (R_X^S)_\infty\}$$

Example

$$S = \{ac \Rightarrow d, e \Rightarrow a, bd \Rightarrow f\}$$
$$X = ce$$
$$Y = \varphi_S(X) = acde$$

$$R_S = \{acd \rightarrow ac, ae \rightarrow e, bdf \rightarrow bd\}$$
$$R_X = \{c \rightarrow true, e \rightarrow true\}$$

$$(R^S_X)_\infty = \{c \rightarrow true, e \rightarrow true, a \rightarrow true, d \rightarrow true, bf \rightarrow b\}$$
$$R_Y = \{a \rightarrow true, c \rightarrow true, d \rightarrow true, e \rightarrow true\}$$

A notion of optimality based on size

Definition: S is *optimal* if
for all equivalent implicational system S'

$$|\,S\,| \leq |\,S'\,|$$

where

$$|\,S\,| = \sum_{A \Rightarrow_S B} |A| + |B|$$

$D(S)$: *direct-optimal* implicational system equivalent to S
Characterized by four necessary and sufficient properties

[Karell Bertet and Mirabelle Nebut 2004]

Optimization rules I

Premise: for all $A \Rightarrow_{D(S)} B$ and $A \Rightarrow_{D(S)} B'$, $B = B'$;

$$\frac{A \Rightarrow B,\ A \Rightarrow C}{\overline{A \Rightarrow BC}}$$

Isotony: for all $A \Rightarrow_{D(S)} B$ and $C \Rightarrow_{D(S)} D$, if $C \subset A$, then $B \cap D = \emptyset$;

$$\frac{A \Rightarrow B,\ AD \Rightarrow BE}{\overline{A \Rightarrow B,\ AD \Rightarrow E}}$$

Optimization rules II

Extensiveness: for all $A \Rightarrow_{D(S)} B$, $A \cap B = \emptyset$;

$$\frac{AC \Rightarrow BC}{AC \Rightarrow B}$$

Definiteness: for all $A \Rightarrow_{D(S)} B$, $B \neq \emptyset$;

$$\frac{A \Rightarrow \emptyset}{\quad}$$

Example

$S = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow a\}$
$I(S) = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow a, e \Rightarrow b, ce \Rightarrow d\}$
$e \Rightarrow b$ by implicational overlap of $e \Rightarrow a$ and $a \Rightarrow b$
$ce \Rightarrow d$ by implicational overlap of $e \Rightarrow a$ and $ac \Rightarrow d$
Optimization: replace $e \Rightarrow a$ and $e \Rightarrow b$ by $e \Rightarrow ab$ (*Premise*)
$D(S) = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow ab, ce \Rightarrow d\}$

$R_S = \{ab \rightarrow a, acd \rightarrow ac, ae \rightarrow e\}$
$(R_S)_\infty = \{ab \rightarrow a, acd \rightarrow ac, ae \rightarrow e, be \rightarrow e, cde \rightarrow ce\}$
$abe \rightarrow e$ (corresponding to $e \Rightarrow ab$): *redundant* in $(R_S)_\infty$

Reason: different underlying proof orderings

Optimize system's size: $|\{e \Rightarrow ab\}| = 3 < 4 = |\{e \Rightarrow a, e \Rightarrow b\}|$

Measure proof of a from X and S:
consider all $B \Rightarrow_S aC$ such that $B \subseteq X$
take multiset of pairs $\langle |B|, \#_B S \rangle$
where $\#_B S$ is number of implications with antecedent B.
Proof of a from $X = \{e\}$ and $\{e \Rightarrow a, e \Rightarrow b\}$: $\{\!\{\langle 1,2\rangle, \langle 1,2\rangle\}\!\}$
Proof of a from $X = \{e\}$ and $\{e \Rightarrow ab\}$: $\{\!\{\langle 1,1\rangle\}\!\}$: *smaller*

Completion optimizes w.r.t. $\prec$:
$\{\!\{\{\!\{ae, e\}\!\}, \{\!\{be, e\}\!\}\}\!\} \prec \{\!\{\{\!\{abe, e\}\!\}\}\!\}$

Rewrite-optimality

Intuition: count symbols in antecedents only once

Definition: S is *rewrite-optimal* if
for all equivalent implicational system S'

$$\parallel S \parallel \leq \parallel S' \parallel$$

where

$$\parallel S \parallel = |Ante(S)| + |Cons(S)|$$

$Ante(S) = \{c : c \in A,\ A \Rightarrow_S B\}$: *set* of symbols in antecedents
$Cons(S) = \{\!\{c : c \in B,\ A \Rightarrow_S B\}\!\}$: *multiset* of symbols in consequents

Example revisited: proof ordering for rewrite-optimality

$S = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow a\}$
$\|\{e \Rightarrow ab\}\| = 3 = \|\{e \Rightarrow a, e \Rightarrow b\}\|$
Replacing $\{e \Rightarrow a, e \Rightarrow b\}$ by $\{e \Rightarrow ab\}$ no longer justified:
$D(S) = I(S)$
associated rewrite system is $(R_S)_\infty$

Measure proof of a from X and S:
consider all $B \Rightarrow_S aC$ such that $B \subseteq X$
take *set* of cardinalities $|B|$
Proof of a from $X = \{e\}$ and $\{e \Rightarrow a, e \Rightarrow b\}$: $\{\!\{1\}\!\}$
Proof of a from $X = \{e\}$ and $\{e \Rightarrow ab\}$: $\{\!\{1\}\!\}$: *equal*

Canonical system

Intuition: omit *Premise* rule (natural for Horn!)

Definition: Given S, the *canonical* implicational system $O(S)$ generated from S is the closure of S with respect to implicational overlap, isotony, extensiveness and definiteness.

Completion by $\leadsto_O$ generates canonical system

$\leadsto_O$: deduction mechanism with implicational overlap (expansion) and *isotony*, *extensiveness* and *definiteness* (contraction)

Proposition: Given implicational system S for all fair and contracting derivations

$$S = S_0 \leadsto_O S_1 \leadsto_O \cdots$$

we have

$$S_\infty = O(S)$$

Correspondence of $\leadsto_O$ and $\leadsto_R$

Intuition: every step by isotony, extensiveness and definiteness is covered by simplification and deletion.

Lemma: For all implicational systems S

$$\text{if } S \leadsto_O S' \text{ then } R_S \leadsto_R R_{S'}$$

Deduction mechanisms correspond up to redundancy

Intuition: whatever is generated by $\leadsto_O$ is generated by $\leadsto_R$ but may become redundant eventually.
Theorem: For every implicational system S, for all fair and contracting derivations

$$S = S_0 \leadsto_O S_1 \leadsto_O \cdots$$

and

$$R_S = R_0 \leadsto_R R_1 \leadsto_R \cdots$$

for all $FG \rightarrow F \in R_{(S_\infty)}$:
either $FG \rightarrow F \in (R_S)_\infty$ or $FG \rightarrow F$ is redundant in $(R_S)_\infty$.

Summary and directions for future work

- Implicational systems, Moore families, forward chaining, implicational overlap, direct system, direct-optimal system
- Rewriting-based framework:
  - Generate direct system by equational overlap
  - Compute minimal models
  - Rewrite-optimal system
  - Generate rewrite-optimal system by equational overlap and simplification
- Future: investigations of canonicity in more general theories