On a rewriting approach to satisfiability procedures: extension, combination of theories and an experimental appraisal

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[1] Joint work with Alessandro Armando, Silvio Ranise, and Stephan Schulz

- Motivation
  - The big picture
  - Decision procedures
  - Little engines and big engines of proof
- Rewrite-based satisfiability: new results
  - A rewrite-based methodology for T-satisfiability
  - Theories of data structures
  - A modularity theorem for combination of theories
- Experimental appraisal
  - Comparison of E with CVC and CVC Lite
  - Synthetic benchmarks (valid and invalid): evaluate scalability
  - “Real-world” problems
- Summary

Certification of traditional systems (e.g., airplane wing)

- Build mathematical models (e.g., sets of differential equations) of the design, its environment, and requirements
- Use calculation to establish that the design in the context of the environment satisfies the requirements
- Only useful when *mechanized*
- Models are validated by testing
- Limited testing suffice because we are dealing with continuous systems
- This is *product-based* certification

Certification of software systems

- Mostly done by controlling, monitoring and documenting the process of software creation
- This is *process-based* certification
- Testing is product-based but not sufficient because we are dealing with discrete systems:
    - Complete testing is unfeasible
    - Extrapolation from incomplete tests unjustified

Product-based certification for software

- Build mathematical models of the design, its environment, and requirements
    - The "applied math" of Computer Science is formal logic
    - Models are formal descriptions in some logical systems
- Use calculation to establish that the design in the context of the environment satisfies the requirements
    - Calculation in formal logic is done by *theorem proving* or *model checking*:

      $$\mathit{assumptions} \; + \; \mathit{design} \; + \; \mathit{environment} \vdash \mathit{requirements}$$

      It can cover all modeled behaviors, even if numerous or infinite (the power of symbolic reasoning)
- Only useful when *mechanized*
    - So need **automated** theorem proving or model checking

However ...

- Formal calculations
  - Are undecidable in general
  - Even decidable problems have much greater computational complexity than mechanizations of continuous mathematics
- So full automation is impossible in general: need to
  - Rely on heuristics which will sometimes fail: *automated theorem proving with heuristic search*
  - Rely on human guidance: *interactive theorem proving*
  - Trade-off accuracy or completeness of the model for tractability and automation of calculation: *model checking*

Current practice

- Model checking used to look for errors (debugging)
- Verification (show the absence of errors) much less practiced
- **Challenges**:
    - Make model checking useful for verification
    - Make relevant theorem proving automated
    - Make model checking and theorem proving work together

Research context

- Model checking requires simple models (e.g., finite state)
  - But can be used to verify properties of a complex model if it has *property-preserving* abstraction
  - "Abstract-check-refine" paradigm
  - **First key idea**: use theorem proving to calculate the abstraction
- Classical verification poses correctness as a single "big theorem": failure to prove it (if true) means disaster
  - **Second key idea**: "fault-tolerant" theorem proving:
  - Prove lots of small theorems instead of a big one
  - In a context where some failures can be tolerated
- Automated abstraction provides precisely such a context!

Decision procedures

This notion of theorem proving is based on powerful decision procedures:

- Reasoning about software requires reasoning about theories of data types, e.g., lists, arrays, integers, trees, tuples or records, sets, reals.
- Some of these theories or fragments thereof are *decidable*.
- *Decision procedures* to be embedded in verification tools and proof assistants, interfaced with model checkers.

Decision procedure for T-satisfiability

An algorithm that takes in input a set S of *ground* T-literals and reports:

- *unsatisfiable* if no T-model satisfies S,
- *satisfiable* otherwise (should return the model as well).

If such an algorithm exists, T-satisfiability is decidable.

Problems that reduce to T-(un)satisfiability

Decision procedures do not handle quantifiers: either the problem is *ground* (i.e., no variables) or there are only $\forall$-quantified variables that are eliminated through negation and Skolemization:

- Word Problem: $T \models s \simeq t$, if $S = \{s \not\simeq t\}$ is T-unsat.
- Uniform Word Problem: $T \models \bigwedge_{i=1}^{n} p_i \simeq q_i \supset s \simeq t$, if $S = \{p_1 \simeq q_1, \ldots p_n \simeq q_n, s \not\simeq t\}$ is T-unsat.
- Clausal Validity Problem: $T \models \bigwedge_{i=1}^{n} p_i \simeq q_i \supset \bigvee_{j=1}^{m} s_j \simeq t_j$, if $\{p_1 \simeq q_1, \ldots p_n \simeq q_n, s_1 \not\simeq t_1, \ldots s_m \not\simeq t_m\}$ is T-unsat.
- $T \models \varphi$ (arbitrary formula), if each conjunction of literals from $DNF(\neg\varphi)$ is T-unsat (not practical if DNF is generated explicitly).
- S is T-sat: model is counter-example to original conjecture.

Example of set of literals

$$x \leq y, \qquad y \leq x + z$$

$$p(x - y) \simeq \mathit{true}, \qquad p(z - y) \simeq \mathit{true}, \qquad p(0) \simeq \mathit{false}$$

$$\mathit{select}(\mathit{store}(v, i, 0), j) \simeq z, \qquad \mathit{select}(v, j) \simeq y$$

combines:

- the theory of equality with *free* (*uninterpreted*) function symbols (e.g., p), and
- integer arithmetic with *defined* (*interpreted*) function symbols (e.g., $+, -, \leq$), and
- the theory of arrays, where $\mathit{select}, \mathit{store}$ are *defined* (*interpreted*) function symbols.

Little engines of proof I

Design, prove sound and complete, and implement a satisfiability procedure for *each theory*, e.g.:

- Theory of equality with free symbols: *congruence closure* [Kozen 1977; Shostak 1978; Downey-Sethi-Tarjan 1980]
- Theory of lists: congruence closure with *axioms built-in* [Nelson-Oppen 1980; Shostak 1984]
- Theory of arrays with extensionality: congruence closure with *pre-processing wrt axioms* and case analysis [Stump-Barrett-Dill-Levitt 2001]

Little engines of proof II

Combination of theories [Nelson-Oppen 1979; Shostak 1984]:

$$T_1, \dots T_n$$

- $\mathcal{T}_i$'s don't share function symbols: if a T_i-term r occurs under a T_j symbol f, rename as x (new var) and add $x \simeq r$ (e.g., $c \simeq 2 + car(l)$ becomes $c \simeq 2 + x, x \simeq car(l)$)
- Communication among procedures: only equalities between variables
- Complete for *convex* theories: if $T \models \Gamma \supset \bigvee_{j=1}^{m} s_j \simeq t_j$, then $T \models \Gamma \supset s_j \simeq t_j$ for some j, where Γ is a conjunction of equalities

Little engines of proof III

- Equality with free function symbols is convex:
  if a disjunction of ground equalities is valid, one is valid
- Linear arithmetic is convex if there are only equalities, not with disequalities:
  - $\mathcal{LA}(Q)$: $x \leq y \vee y \leq x$ is valid but neither disjunct is.
  - $\mathcal{LA}(Z)$: $1 \leq x \wedge x \leq 2 \supset x \simeq 1 \vee x \simeq 2$ is valid but neither $1 \leq x \wedge x \leq 2 \supset x \simeq 1$ nor $1 \leq x \wedge x \leq 2 \supset x \simeq 2$ is.
- The theory of arrays is not convex:
  $i \simeq j \vee \mathit{select}(\mathit{store}(a, i, v), j) \simeq \mathit{select}(a, j)$ is valid but neither disjunct is.
- Non-convex: case analysis or “splitting” (in practice: backtracking): non-deterministic

Big engines of proof (in brief)

Methods for theorem proving in first-order logic with equality:

- Herbrand theorem (1930): unsatisfiability is semi-decidable
- Ordering-based methods: resolution, hyperresolution, subsumption, paramodulation/superposition, simplification
- Non-deterministic: combine with *fair* search plan to get deterministic semi-decision procedure
- Any first-order theory T, any conjecture φ: $T \cup \{\neg\varphi\} \vdash^{?} \bot$
- May have theories built-in (equality for sure) (e.g., AC)

Issues with little engines

- Combination of theories:
  done by combining procedures rather than theories:
  complicated, *ad hoc*
- Soundness and completeness proof: if given, is *ad hoc*
- Implementation: usually from scratch: correctness?
  integration in different environments? duplicated work?
- **Challenge**: can we get something good for decision procedures from big engines?

From a big-engine perspective

- Combination of theories: give union of presentations as input to prover
- Soundness and completeness proof: given once and for all for first-order inference system
- Implementation: (re-)use first-order prover (techniques, code)
- Proof generation: already there by default
- Model generation: final T-sat set (starting point)
- **Key issue**: prove termination

Decision procedures: summary

- **Objective**: Decision procedures for application of automated reasoning to verification
- **Desiderata**: Fast, expressive, easy to use, extend, integrate, prove sound and complete
- **Issues**:
  - Combination of theories:
    usually done by combining procedures: complicated? *ad hoc*?
  - Soundness and completeness proof: usually *ad hoc*
  - Implementation: usually from scratch: correctness? integration in different environments? duplicated work?

“Little” engines and “big” engines of proof: summary

- “Little” engines, e.g., validity checkers for specific theories
  Built-in theory, quantifier-free conjecture, decidable
- “Big” engines, e.g., general first-order theorem provers
  Any first-order theory, any conjecture, semi-decidable
- Not an issue of size (e.g., lines of code) of systems!
- Continuity: e.g., “big” engines may have theories built-in
- **Challenge**: can we get something good for decision procedures from big engines?

What kind of theorem prover?

First-order logic with equality

$\mathcal{SP}$ inference system: rewrite-based

- *Simplification by equations*: normalize clauses
- *Superposition*: generate clauses

Complete simplification ordering (CSO) $\succ$ on terms, literals and clauses: $\mathcal{SP}_\succ$

(Fair) $\mathcal{SP}_\succ$*-strategy* : $\mathcal{SP}_\succ$ $+$ (fair) search plan

Rewrite-based methodology for T-satisfiability

- *T-satisfiability*: decide satisfiability of set S of ground literals in theory (or combination) T
- *Methodology*:
    - *T-reduction*: apply inferences (e.g., to remove certain literals or symbols) to get equisatisfiable *T-reduced* problem
    - *Flattening*: flatten all ground literals (by introducing new constants) to get equisatisfiable T-reduced *flat* problem
    - *Ordering selection and termination*: prove that any fair $\mathcal{SP}_{\succ}$-strategy terminates when applied to a $\mathcal{T}$-reduced flat problem, provided $\succ$ is *$\mathcal{T}$-good*
- Everything *fully automated* except for termination proof

Covered theories

- *EUF*, *lists*, *arrays* with and without extensionality, *sets* with extensionality [Armando, Ranise, Rusinowitch 2003]
- *Records* with and without extensionality, *integer offsets*, *integer offsets modulo* [Armando, Bonacina, Ranise, Schulz 2005]

In experiments: arrays, records, integer offsets, integer offsets modulo, EUF and combinations

Records: presentation

Sort $\mathrm{REC}(id_1 : \mathrm{T}_1, \ldots, id_n : \mathrm{T}_n)$

$$\begin{array}{lll} \forall x, v. & \mathsf{rselect}_i(\mathsf{rstore}_i(x, v)) \simeq v & 1 \leq i \leq n \\ \forall x, v. & \mathsf{rselect}_j(\mathsf{rstore}_i(x, v)) \simeq \mathsf{rselect}_j(x) & 1 \leq i \neq j \leq n \\ \forall x, y. & (\bigwedge_{i=1}^{n} \mathsf{rselect}_i(x) \simeq \mathsf{rselect}_i(y) \supset x \simeq y) & \end{array}$$

where x, y have sort REC and v has sort T_i.
Extensionality is the third axiom.

Records: termination of $\mathcal{SP}$

$\mathcal{R}$*-reduction*: eliminate disequalities between records by resolution with extensionality + splitting.

$\mathcal{R}$*-good*: $t \succ c$ for all ground compound terms t and constants c.

Termination: case analysis of generated clauses (CSO plays key role).

Theorem: A fair $\mathcal{R}$-good $\mathcal{SP}_{\succ}$-strategy is a satisfiability procedure for the theories of records and records with extensionality.

Integer offsets: presentation

A fragment of the theory of the integers:

s: successor

p: predecessor

$$\begin{array}{ll} \forall x. & \mathrm{s}(\mathrm{p}(x)) \simeq x \\ \forall x. & \mathrm{p}(\mathrm{s}(x)) \simeq x \\ \forall x. & \mathrm{s}^i(x) \not\simeq x \quad \text{for } i > 0 \end{array}$$

Infinitely many *acyclicity axioms*!

Integer offsets: termination of $\mathcal{SP}$

$\mathcal{I}$-reduction: eliminate p by replacing $\mathrm{p}(c) \simeq d$ with $c \simeq \mathrm{s}(d)$:
first two axioms no longer needed.
Bound the number of acyclicity axioms:
$\forall x.\, \mathrm{s}^i(x) \not\simeq x$ for $0 < i \leq n+1$
if there are n occurrences of s.

$\mathcal{I}$-good: any CSO.
Termination: case analysis of generated clauses.

Theorem: A fair $\mathcal{SP}_{\succ}$-strategy is a satisfiability procedure for the theory of integer offsets.

Integer offsets modulo: presentation

To reason with indices ranging over the integers mod k ($k > 0$):

$$\begin{array}{lll}
\forall x. & \mathrm{s}(\mathrm{p}(x)) \simeq x & \\
\forall x. & \mathrm{p}(\mathrm{s}(x)) \simeq x & \\
\forall x. & \mathrm{s}^i(x) \not\simeq x & 1 \leq i \leq k-1 \\
\forall x. & \mathrm{s}^k(x) \simeq x &
\end{array}$$

Finitely many axioms.

Integer offsets modulo: termination of $\mathcal{SP}$

$\mathcal{I}$-reduction: same as above.

$\mathcal{I}$-good: any CSO.

Termination: case analysis of generated clauses.

Theorem: A fair $\mathcal{SP}_{\succ}$-strategy is a satisfiability procedure for the theory of integer offsets modulo.

Termination also without $\mathcal{I}$-reduction.

A modularity theorem for combination of theories

- *Modularity*: if $\mathcal{SP}_{\succ}$-strategy decides $\mathcal{T}_i$-sat problems then it decides $\mathcal{T}$-sat problems, $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i$
- $\mathcal{T}_i$-reduction and flattening apply as for each theory
- Termination?

Three simple conditions

- $\succ$ $\mathcal{T}$-*good*, if $\mathcal{T}_i$-*good* for all i, $1 \leq i \leq n$
- The $\mathcal{T}_i$ do not share function symbols
  (*Intuition*: no superposition from compound terms across theories)
- Each $\mathcal{T}_i$ is *variable-inactive*:
  no maximal literal in a ground instance of a clause is instance of an equation $t \simeq x$ where $x \notin Var(t)$
  (*Intuition*: no superposition from variables across theories, since for $t \simeq x$ where $x \in Var(t)$, $t \succ x$)

A modularity theorem

Theorem: if

- No shared function symbol (shared constants allowed),
- Variable-inactive presentations $\mathcal{T}_i$, $1 \leq i \leq n$,
- Fair $\mathcal{T}_i$-good $\mathcal{SP}_{\succ}$-strategy is satisfiability procedure for $\mathcal{T}_i$,

then
a fair $\mathcal{T}$-good $\mathcal{SP}_{\succ}$-strategy is a satisfiability procedure for $\mathcal{T}$.

EUF, *arrays* (with or without extensionality), *records* (with or without extensionality), *integer offsets* and *integer offsets modulo*, all satisfy these hypotheses.

Two remarks on generality

- *Purely equational theories*:
  no trivial models ⇒ variable-inactive
- *First-order theories*: variable-inactive excludes, e.g.,
  $a_1 \simeq x \vee \ldots \vee a_n \simeq x$, a_i constants (*)
  Such a clause means *not stably-infinite*, hence *not convex* under the no trivial models hypothesis:
  if $\mathcal{T}_i$ not variable-inactive for (*), Nelson-Oppen does not apply either.

Experimental setting

- Three systems:
  - The E theorem prover: E 0.82 [Schulz 2002]
  - CVC 1.0a [Stump, Barrett and Dill 2002]
  - CVC Lite 1.1.0 [Barrett and Berezin 2004]
- Generator of pseudo-random instances of synthetic benchmarks
- 3.00GHz 512MB RAM Pentium 4 PC: max 150 sec and 256 MB per run
- **Folklore**: systems with built-in theories are out of reach for prover with presentation as input ...

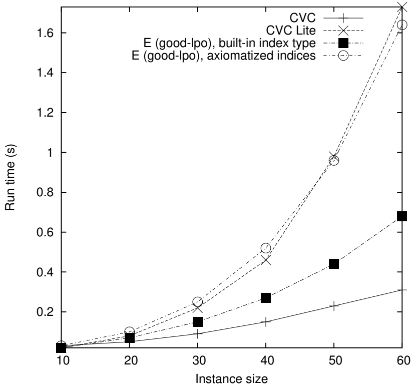
Synthetic benchmarks

- `STORECOMM`(n), `SWAP`(n), `STOREINV`(n): arrays with extensionality
- `IOS`(n): arrays and integer offsets
- `QUEUE`(n): records, arrays, integer offsets
- `CIRCULAR_QUEUE`(n, k): records, arrays, integer offsets mod k

`STORECOMM`(n), `SWAP`(n), `STOREINV`(n): both valid and invalid instances

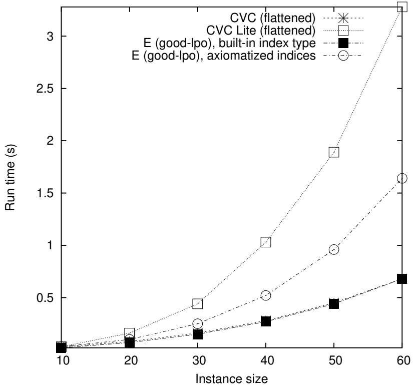
Parameter n: test *scalability*

Performances on valid $\texttt{STORECOMM}(n)$ instances



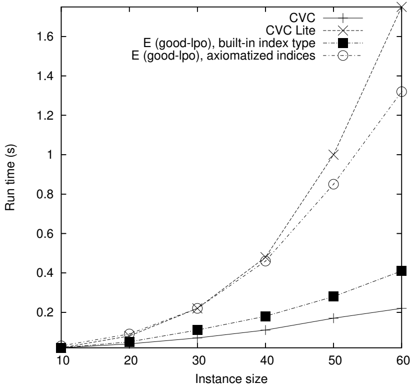
Native input: CVC wins but E better than CVC Lite

Performances on valid STORECOMM(n) instances



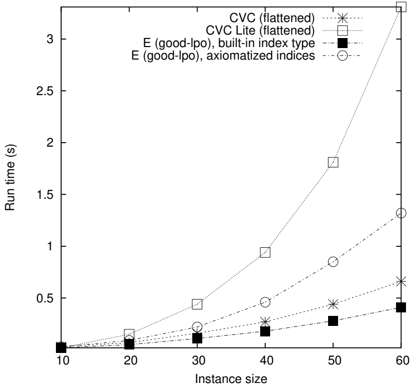
Flat input: E matches CVC

Performances on invalid `STORECOMM`(n) instances



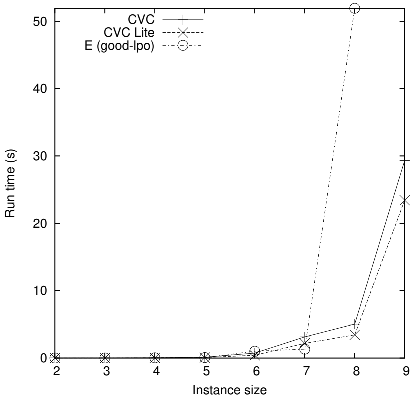
Native input: prover conceived for unsat handles sat even better

Performances on invalid `STORECOMM`(n) instances



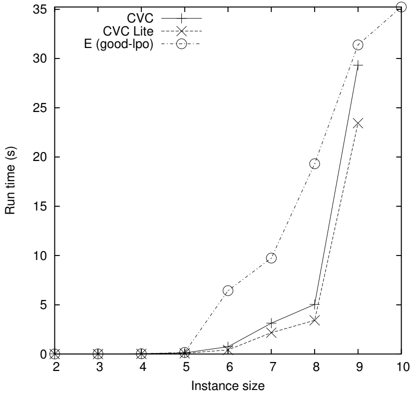
Flat input: E surpasses CVC

Performances on valid `SWAP`(n) instances



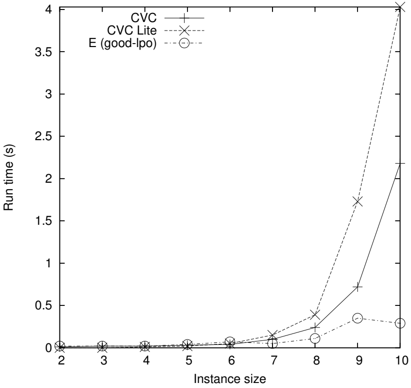
Harder problem: no system terminates for $n \geq 10$

Performances on valid `SWAP`(n) instances



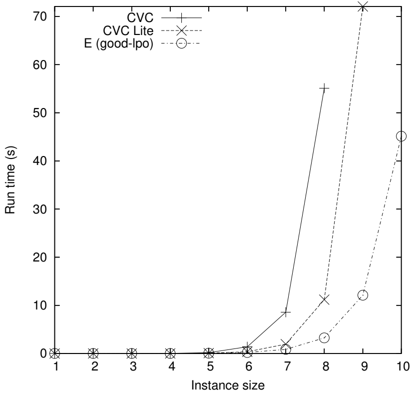
Added lemma for E: additional flexibility for the prover

Performances on invalid $\mathtt{SWAP}(n)$ instances



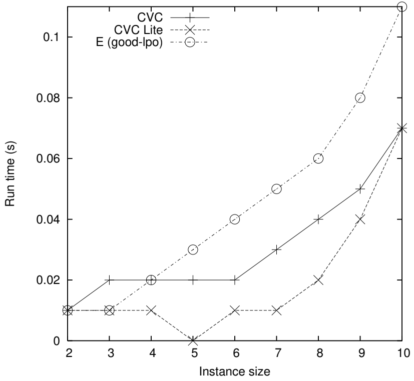
Easier problem, but E clearly ahead

Performances on valid `STOREINV`(n) instances



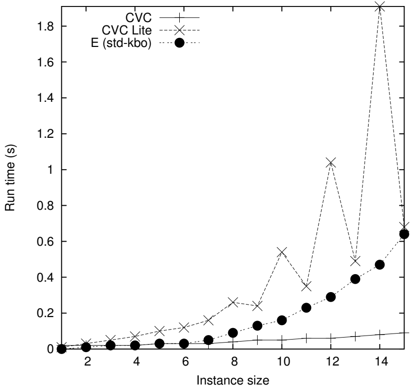
E(std-kbo) does it in *nearly constant time*!

Performances on invalid `STOREINV`(n) instances



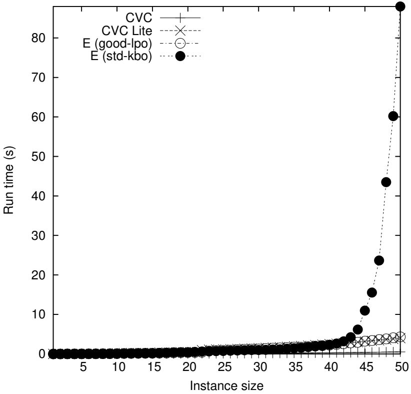
Not as good for E but run times are minimal

Performances on `IOS` instances



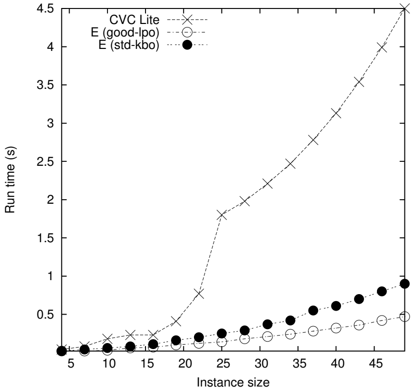
CVC and CVC Lite have built-in $\mathcal{LA}(\mathcal{R})$ and $\mathcal{LA}(\mathcal{I})$ respectively!

Performances on `QUEUE` instances (plain queues)



CVC wins (built-in arithmetic!) but E matches CVC Lite

Performances on $\mathtt{CIRCULAR\_QUEUE}(n, k)$ instances $k = 3$



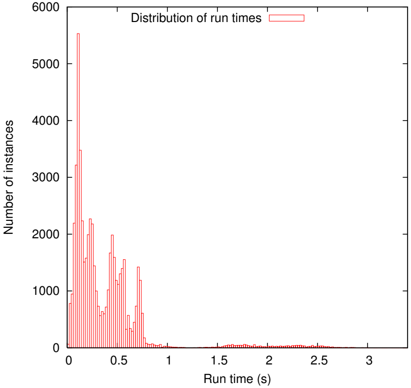
CVC does not handle integers mod k, E clearly wins

"Real-world" problems

- UCLID [Bryant, Lahiri, Seshia 2002]: suite of problems
- haRVey [Déharbe and Ranise 2003]: extract T-sat problems
- over 55,000 proof tasks: integer offsets and equality
- all valid

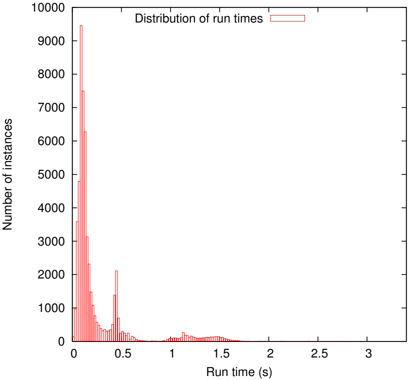
Test performance on huge sets of literals.

Run time distribution for *E(auto)* on UCLID set



Auto mode: prover chooses search plan by itself

Better run time distribution for E on UCLID set



Optimized strategy: found by testing on random sample of 500 problems (less than 1%)

Summary

- *General methodology* for rewrite-based T-sat procedures and its application to several theories of data structures
- *Modularity theorem* for combination of theories
- Experiments: first-order prover
  - *taken off the shelf* and
  - conceived for very different search problems

compares amazingly well with state-of-the-art verification tools

Directions for further research

- Prover's search plans for T-sat problems
- More or stronger termination results
- Precise relationship between variable-inactive and stably-infinite, convex
- Integration with approaches for full $\mathcal{LA}$ or bit-vectors
- $\mathcal{T}$-decision procedures (arbitrary quantifier-free formulæ): integration with SAT-solver? Other approaches?
- Combination with automated model building
- In general: explore "big" engines technology for decision procedures

Big picture

Reasoning environments for verification (and more):

- SAT-solvers
- “Little” engines
- “Big” engines
- Good interfaces
- … … …